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# *DYNAMIC PROGRAMMING AND A NEW FORMALISM IN THE CALCULUS OF VARIATIONS*

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1. *Introduction.*—In a series of papers,[1–11] we have treated a number of mathematical problems arising from multistage decision processes. Problems of this type occur in the theory of probability;[1, 4, 5, 9] in mathematical economics;[1, 2, 6, 7, 8, 11] in control processes;[2, 8] in learning processes;[4, 9] and in many other fields as well.

In this paper we wish to show that the functional-equation technique introduced in the above works may be used to provide a new approach to some classical problems in the calculus of variations. In addition to furnishing a new analytic weapon, we feel that the method has great potentialities as a computational tool. As we have pointed out previously,[1, 3, 4] this approach seems ideally suited to the handling of variational problems involving stochastic processes. This point will be further enlarged upon in some forthcoming publications.

To illustrate the approach in its simplest setting, we shall consider first the problem of maximizing $\int_0^t F(x, z)\, du$, subject to the constraint $dx/du = G(x, z)$, $x(0) =$

$c$, where the analytic detail is at a minimum, and then turn to the eigenvalue problem derived from

$$\frac{d^2x}{dt^2} + \lambda\phi(t)x = 0, \qquad x(0) = x(1) = 0, \tag{1.1}$$

which is rather less straightforward.

In both cases we shall derive a partial differential equation for the desired quantity. The method we sketch below is also applicable to multidimensional eigenvalue problems whenever the underlying space has certain symmetry properties. Further results, together with a detailed exposition of the results contained herein, will appear in another publication.

2. *The Maximization Problem.*—Let us now consider the maximization problem described in the first section. We shall consistently assume that all functions that appear possess all the differentiability properties required to make our operations legitimate, since we are primarily interested here in presenting the basic formalism. The essence of our method lies in considering the maximum value as a function of the parameters which describe the state of the process, the "state variables." In this case these are $c$ and $t$. We write

$$\operatorname*{Max}_{z} \int_0^t F(x, z)\, du = f(c, t). \tag{2.1}$$

The classical approach regards $c$ and $t$ as fixed and considers $\int_0^t F(x, z)\, du$ as a functional of $z$. In contrast, we regard $c$ and $t$ as parameters and seek to determine $z(0)$ as a function of $c$ and $t$.

Speaking in terms of a multistage process, in place of determining the optimal continuation from one fixed position, we try to find the optimal first step from any position.

Let us proceed to the analysis which will clarify the above general remarks. We have

$$f(c, s + t) = \operatorname*{Max}_{z} \int_0^{s+t} F(x, z)\, du = \operatorname*{Max}_{z} \left[\int_0^s F(x, z)\, du + \int_s^{s+t} F(x, z)\, du\right]. \tag{2.2}$$

It is clear that whatever the choice of $z(u)$ in $0 \le u \le s$, if we wish to maximize the right side of (2.2) we must choose $z(u)$ in $s \le u \le s + t$ so as to maximize $\int_s^{s+t} F(x, z)\, du$ subject to the constraints $dx/dt = G(x, z), x(s) = c(s)$. Here $c(s)$ is the new initial value obtained from the differential equation $dx/dt = G(x, z)$, $x(0) = c$.

In view of the invariant nature of the problem with respect to translations, we see that we must have

$$\int_s^{s+t} F(x, z)\, du = f[c(s), t]. \tag{2.3}$$

Hence (2.2) yields the fundamental functional equation

$$f(c, s + t) = \operatorname*{Max}_{z\,[0,\, s]} \left\{\int_0^s F(x, z)\, du + f[c(s), t]\right\}. \tag{2.4}$$

The notation $z[0, s]$ indicates that the maximization is over all $z(u)$ taken over

$0 \leq u \leq s$. As $s$ shrinks to 0, the choice of a function reduces to a choice of $z(0)$, a quantity which we shall call $w$ for typographical convenience. A straightforward calculation yields

$$f_t = \operatorname*{Max}_{w} [F(c, w) + G(c, w)f_c] \tag{2.5}$$

as the limiting form of (2.4) as $s \to 0$. The maximizing $w$ is obtained by equating the partial derivative with respect to $w$ to zero, $0 = F_w + G_w f_c$.[12] Combining this with (2.5), we obtain the system of equations

$$\begin{aligned} f_t &= (FG_w - GF_w)/G_w = K(c, w), \\ f_c &= -F_w/G_w = L(c, w). \end{aligned} \tag{2.6}$$

To derive a partial differential equation for $w$, we employ the identity $(f_t)_c = (f_c)_t$. The result is

$$K_c + K_w w_c = L_w w_t. \tag{2.7}$$

As is well known,[13] the general solution of this equation may be obtained from the general solution of a system of first-order ordinary differential equations. Since $c$ is known for $t = 0$, to complete the solution we require only the value of $w$ at $t = 0$. For small $t$ we have

$$\int_0^t F(x, z)\, du = F(c, w)t + 0(t^2). \tag{2.8}$$

Hence $w$ at $t = 0$ is determined by $F_w = 0$.

Taking $G(x, z) = z$, we are reduced to the familiar problem of maximizing $\int_0^t F(x, x')\, du$. It is readily verified that the characteristic equations obtained from (2.7) are equivalent to the usual Euler equation.

The general problem of maximizing $\int_0^t F(x_1, x_2, \ldots, x_n; z_1, z_2, \ldots, z_m)du$ subject to the constraints $dx_i/dt = G_i(x, z)$, $x_i(0) = c_i$ may be attacked in the same fashion.

3. *The Eigenvalue Problem.*—The eigenvalue problem posed in §1 is equivalent to that of determining the successive minima of $\int_0^1 x'^2\, du$ subject to the constraints $\int_0^1 \phi x^2\, du$, $x(0) = x(1) = 0$. To treat the problem using the functional-equation approach, we imbed it within the more general problem of determining the minima of $J(x) = \int_a^{a+t} x'^2\, du$ subject to the constraints

$$\begin{aligned} &(a) \quad x(a) = x(a + t) = 0, \\ &(b) \quad \int_a^{a+t} \phi(u)x^2\, du + k \int_a^{a+t} \phi(u)\,(a + t - u)x\, du = 1. \end{aligned} \tag{3.1}$$

Let Min $J(u) = f(a, k, t)$. Then, if, as above, we write

$$f(a, k, s + t) = \int_a^{a+s} x'^2\, du + \int_{a+s}^{a+s+t} x'^2\, du, \tag{3.2}$$

we cannot derive a functional equation immediately, since $x(a + s)$ is not necessarily zero. To simplify the analysis, let us pass directly to the derivation of the partial differential equation, assuming that $s$ is an infinitesimal. It is sufficient to consider only terms which are of zeroth or first order in $s$. Since $x(a + s) = sx'(a) + \ldots$, we set $y(u) = x(u) - sx'(a)\,(a + s + t - u)/t$. Then $y(a + s) =$

$y(a + s + t) = 0$, to the order of our approximation, and $x'(u) = y'(u) - sx'(a)/t$. Thus, to terms in $s^2$,

$$\int_{a+s}^{a+s+t} x'^2\, du = \int_{a+s}^{a+s+t} y'^2\, du.$$

The constraint of (3.1*b*) becomes

$$\int_{a+s}^{a+s+t} \phi(u) y^2\, du + \left[k + \frac{2sx'(a)}{t}\right] \int_{a+s}^{a+s+t} \phi(u) y(u)(a + s + t - u)\, du = 1 - s\psi[k, x'(a), t] + 0(s^2), \quad (3.3)$$

where

$$\psi[k, x'(a), t] = \frac{kx'(a)}{t} \int_a^{a+t} \phi(u)\, (a + t - u)^2\, du. \quad (3.4)$$

Making a change of variable $y = (1 - (s\psi/2))w$, to reduce the constant term in (3.3) to 1, the constraint takes the form

$$\int_{a+s}^{a+s+t} \phi(u) w^2\, du + \left\{k + s\left[\frac{k\psi}{2} + \frac{2x'(a)}{t}\right]\right\} \int_{a+s}^{a+s+t} \phi(u) w(u)\, (a + s + t - u)\, du = 1 + 0(s^2). \quad (3.5)$$

The equation in (3.2) becomes

$$f(a, k, s + t) = \min_{x'[a,\, a+s]} \left\{\int_a^{a+s} x'^2\, du + (1 - s\psi) f\left[a + s, k + s\left(\frac{k\psi}{2} + \frac{2x'[a]}{t}\right), t\right]\right\} + 0(s^2). \quad (3.6)$$

Letting $s \to 0$, this yields the partial differential equation

$$f_t = \operatorname*{Min}_{x'(a)} \left\{x'(a)^2 + f_a + \left[\frac{k\psi}{2} + \frac{2x'(a)}{t}\right] f_k - \psi f\right\}. \quad (3.7)$$

Since $\psi = x'(a)\Phi$ is independent of $x'(a)$, the unique minimum occurs when

$$2x'(a) + \left(\frac{k\Phi}{2} + \frac{2}{t}\right) f_k - \Phi f = 0. \quad (3.8)$$

Eliminating $x'(a)$ by means of this last relation, we obtain a nonlinear partial differential equation for $f$.

The behavior of $f$ for small $t$ may be determined by solving the problem of minimizing $\int_a^{a+t} x'^2\, du$ subject to the constraints

$$x(a) = x(a + t) = 0, \quad (3.9)$$

$$\phi(a) \int_a^{a+t} x^2\, du + k \int_a^{a+t} \phi(u)\, (a + t - u) x\, du = 1.$$

This problem, in turn, may be treated by the partial differential equation approach, using the fact that the solution for $k = 0$ is immediate.

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# A CHARACTERIZATION OF TAME CURVES IN THREE-SPACE

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1. *Introduction.*—A subset $K$ of 3-space has been called tamely imbedded if there is a homeomorphism on space throwing $K$ onto a polyhedron. If $K$ is a simple arc or simple closed curve (i.e., a 1-manifold) that is contained in a plane, it follows quickly from classical results of Schoenflies that $K$ is tamely imbedded. If the 1-manifold is not a subset of a plane, it is necessary to distinguish whether $K$ is an arc or a closed curve. If $K$ is a polygonal arc, it is already tame and also has the property of being equivalent to an interval under a semilinear map on 3-space. If $K$ is a polygonal closed curve, it is already tame, but the intermediate step of mapping $K$ into a plane is neither advantageous nor possible if $K$ is knotted.

About thirty years ago an example was given of an arc $K$ which is not equivalent to a subset of a plane under any homeomorphism on 3-space. This example and others closely related to it, including some simple closed curves, were studied by Antoine,[1] Alexander,[2] and later by Fox and Artin[3] and serve to emphasize the pathological difficulties that may occur. The nature of these examples suggests that the difficulties are of a local nature as regards the manner the 1-manifold is imbedded in 3-space. This point of view is borne out by recent results of R. H. Bing[4] and E. E. Moise,[5] who prove that a locally tamely imbedded set is tamely imbedded.

The purpose of the present paper is to characterize, by means of positional invariants, those 1-manifolds which are locally tamely imbedded. By the above-mentioned result, this implies a characterization of the tamely imbedded 1-mani-