Galton's Quincunx : random walk or chaos?

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Abstract

In 1873 Francis Galton had constructed a simple mechanical device where a ball is dropped vertically through a harrow of pins that deflect the ball sideways as it falls. Galton called the device a quincunx, although today it is usually referred to as a Galton board. Statisticians often employ (conceptually, if not physically) the quincunx to illustrate random walks and the central limit theorem. In particular how a Binomial or Gaussian distribution results from the accumulation of independent random events, that is, the collisions in the case of the quincunx. But how valid is the assumption of "independent random events" made by Galton and countless subsequent statisticians? This paper presents evidence that this assumption is almost certainly not valid and that the quincunx has the richer, more predictable qualities of a low-dimensional deterministic dynamical system. To put this observation into a wider context, the result illustrates that statistical modelling assumptions can obscure more informative dynamics. When such dynamical models are employed they will yield better predictions and forecasts.

1 Introduction

In today's terms Francis Galton (1822–1911) might be described as: explorer, geneticist, meteorologist, statistician. One of his most significant works is the book *Natural Inheritance* (Galton, 1889), which, amongst other things, provides foundations for several important principles of statistics. Galton was not a significant mathematician and used experiments and mechanical devices to both illustrate principles and as tools to attain insights. The *quincunx* is such a mechanical device and was first publicly demonstrated at the Royal Institute in February 1874 (Stigler, 1986). The device consists of two parallel, vertical planes, between which are many horizontal rows of equally spaced pins, with alternate rows offset by half the pin spacing; see figure 1. At the top is a funnel into which lead shot (small spherical balls of lead) is poured, and at the bottom a row of compartments to collect the shot. In Galton's own words (the following quotes are from *Natural Inheritance*), when lead shot is dropped into the device it "scampers deviously down through the pins in a curious and interesting way; each of them darting a step to the right or left, as the case may be, every time it strikes a pin."

One purpose of this device is to illustrate the "Law of dispersion", or central limit theorem, in that the "cascade [of shot] issuing from the funnel broadens as it descends," and, at length, when collected at the compartments at the base, approximates the Binomial or Normal distribution. Galton explains this as follows:

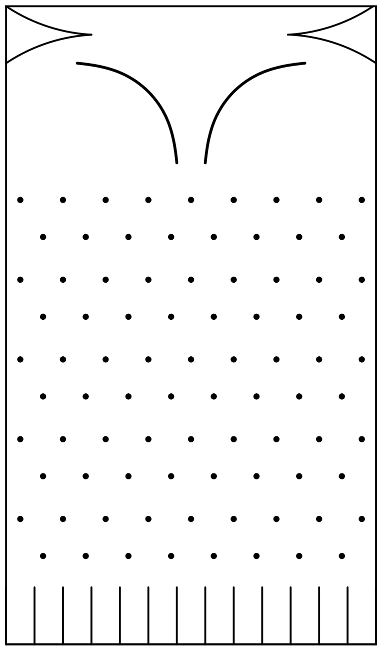

Figure 1: Schematic of Galton's quincunx device.

> The principle on which the action of the apparatus depends is, that a number of small and independent accidents befall each shot in its career. In rare cases, a long run of luck continues to favour the course of a particular shot towards either outside place, but in the large majority of instances the number of accidents that cause Deviation to the right, balance in a greater or less degree those that cause Deviation to the left. Therefore most of the shot finds its way into the compartments that are situated near a perpendicular line drawn from the outlet of the funnel, and the Frequency with which shots stray to different distances to the right or the left of that line diminishes in a much faster ratio than those distances increase.

Many modern statisticians view the quincunx in a similar way, that is, they assume "independent accidents". The internet abounds with so called simulations of the quincunx that are not simulations of the mechanical device, but simply simulations of a Binomial process, or random walk. Some readers might, as we do, question the assumption of "independent accidents" and view the mechanical device as a deterministic dynamical system.

In this article we consider whether a quincunx typically behaves the way Galton and many modern statisticians appear to assume. We will model the quincunx as a deterministic dynamical system. One advantage we have over Galton is that it is now easier to construct a plausible computer simulation than it is to construct a physical quincunx, which means we can simulate quincunx with a variety of physical properties. Doing so we conclude that a quincunx typically does not behave according to Galton's view, that is, that path of the shot is not beset by "independent accidents".

We proceed by first constructing a model of a quincunx, using Galton's original device as a guide for our physical parameters. We then fold this model into a convenient three-dimensional *map*, that is, a discrete-time deterministic dynamical system. This map, which we will call the *quincunx map*, often has complex, apparently chaotic, dynamics with an extensive variety of unstable periodic orbits. For some parameter values the map has a globally attracting stable periodic orbit. (This fact alone begins to cast doubt on Galton's view.) We do not make detailed analysis of the dynamics and bifurcations of the quincunx map. Instead we numerically test the statistical hypothesis that a quincunx device for typical parameter values approximates a Binomial process, and find that it does not. To demonstrate this point we employ a straight-forward analysis of symbolic dynamics of the quincunx map. We find in most cases that the distribution of observed symbols is not consistent with an independence assumption, and even when the distribution is consistent, the symbol sequence is better modelled by a higher order Markov process than a simple random walk.

2 Modelling the quincunx

We attempt to construct a plausible model of Galton's quincunx device. We will state our assumptions as we proceed, and indicate how the model might be improved. Numerical simulations of the Galton board has previously arisen as models of percolation and transport phenomena. Our simulation is similar to some earlier simulations of Galton boards and Lorentz gases (Masliyah and Bridgewater, 1974; Hoover and Moran, 1992; Bruno et al., 2003), except we design our model and select parameters to closely model the quincunx device Galton had constructed in 1873, which is now part of the Galton Collection at the University College London Museum. Our model is based on Galton's writings and photographs of the extant device.

In Galton's description of his quincunx he states there is "about a quarter of an inch" ($\approx$ 0.5cm) between the back board and the front glass sheet. The lead shot Galton used appears to be only slightly smaller than the gap between the planes. Hence, we ignore the three-dimensional structure and consider dynamics in a two dimensional plane.

We assume that the pins of the quincunx harrow are arranged in horizontal rows with pins equally spaced a distance H apart, rows spaced a distance V apart vertically, and every other row offset horizontally by $H/2$. Each pin then has four neighbours, the five pins forming a cross like the dots on the five face of a dice. This cross arrangement is called a *quincunx*, from which Galton's device derives its name. Galton appears to choose $V/H = \sqrt{3}/2$ so that adjacent pins form equilateral triangles, and H around half an inch ($\approx$ 1cm). Throughout the following we use $H = 1$cm and $V = \sqrt{3}/2$cm.

We assume the lead shot is spherical and the pins are cylindrical, so that impact between a shot and pin occurs when the centre of the shot is a distance R from the centre of a pin. Given the above measurements $R \approx 0.25$cm for Galton's quincunx, but we will consider the range $0.2 \leq R \leq 0.3$.

Modelling impacts is difficult and we choose the following simple *physics assumptions*. We ignore rotational motion of the shot, assume instantaneous non-elastic impacts without skidding that retard the shot in proportion to its impact velocity, and allow that if shot impacts the pin with too little velocity it will stick to the pin then roll off without slipping. We also assume there is no air resistance, or retarding from impacts with the vertical walls, and uniform vertical gravitational acceleration $g = 981\text{cm/s}^2$, so that in free flight between pins the centre of the shot traces a parabolic curve,

$$(x(t), y(t)) = (x_0 + u_0 t, y_0 + v_0 t - \frac{1}{2} g t^2), \tag{1}$$

with instantaneous velocity

$$(u(t), v(t)) = \frac{d}{dt}(x(t), y(t)) = (u_0, v_0 - gt), \tag{2}$$

where (x_0, y_0) is the initial position of the centre of the shot at $t = 0$ and (u_0, v_0) the initial velocity. If a pin is situated at (p, q), then impact occurs when

$$(x(t) - p)^2 + (y(t) - q)^2 = R^2, \tag{3}$$

which we note is a quartic in t, and hence easily solved by closed formulae.

The physics assumptions imply that when a shot impacts a stationary pin the rebound velocity of the shot makes the same angle with the normal at the contact point as the incident velocity. The velocity is reduced by a non-dimensional factor γ, called the *coefficient of restitution*. Hence, if t^* is the time of impact, $z = (x(t^*), y(t^*))^T$, $w = (u(t^*), v(t^*))^T$, and the impact occurs with the pin whose centre is at $r = (p, q)^T$, then the rebound velocity is given by

$$\gamma \left(w - 2 \frac{(z-r)^T w}{(z-r)^T (z-r)} (z-r) \right). \tag{4}$$

Non-elastic impact can result in the shot bouncing repeatedly on a pin with exponentially decreasing velocity. We assume there is a threshold $S = 10^{-3}$cm/s so that if the magnitude of the rebound velocity is less than S, then the shot sticks to the pin, then rolls without slipping until it separates. If θ_0 is the angle from the horizontal through the centre of the pin (p, q) to the sticking point $(x(t^*), y(t^*)) = R(\cos(\theta_0), \sin(\theta_0))$, then it can be shown (Speigel, 1967) that if the shot rolls without slipping, then the point of separation occurs at the angle θ_s and speed V_s given by

$$\sin(\theta_s) = \frac{10}{17} \sin(\theta_0) \quad \text{and} \quad V_s^2 = \frac{10}{17} g R \sin(\theta_0). \tag{5}$$

The separation velocity $(u_s, v_s) = V_s(\sin(\theta_s), -\cos(\theta_s))$ being tangent to the pin at the final point of contact $(x_s, y_s) = R(\cos(\theta_s), \sin(\theta_s))$.

An appropriate value of γ is difficult to determine without physical experiment. Masliyah and Bridgewater (1974) used $\gamma = 0.8$ to avoid stick-and-roll motions in their simulations. Bruno et al. (2003) performed physical experiments with polystyrene disks and in their numerical simulations used $\gamma = 0.8$ following Masliyah and Bridgewater (1974). Our simple experiments suggest small lead weights, perhaps similar to the lead shot Galton used, have significantly lower values of γ, so our experiments consider the range $0.4 \leq \gamma \leq 0.8$.

Numerical simulation with the above assumptions is straight forward for a finite set of pins with centres (p_i, q_i). Iteration of the following two steps is a sufficient algorithm:

1. Given (x_0, y_0) and (u_0, v_0) compute the real zeros of the quartic polynomial (3) for $(p, q) = (p_i, q_i)$ for each i. Let t_i^* be the smallest real and positive solution, setting $t_i^* = \infty$ if no real and positive zeros. If $t_i^* = \infty$ for all i, then the shot has existed the harrow.

2. Let $t^* = \min_i t_i^*$. Compute the rebound velocity by (4) using $(p, q) = (p_i, q_i)$ for which the minimum t^* occurs. If the magnitude of the rebound velocity exceeds S, then set (u_0, v_0) to the rebound velocity and $(x_0, y_0) = (x(t^*), y(t^*))$ from (1). Otherwise, the shot sticks and rolls, in which case use (5) to compute $(x_0, y_0) = (x_s, y_s)$ and $(u_0, v_0) = (u_s, v_s)$.

In step 1 one could try to be more sophisticated about which pins a shot is likely to impact, but numerical solution of quartic polynomials is routine.

Figure 2 shows a typical simulation computed numerically as above, where the curves show the path of the centre of the shot and the circles are of radius R centred on the pins. The parameters used in the computation of figure 2 were deliberately chosen to give what might appear to be an approximately Binomial or Gaussian distribution of exit points from the harrow, however, we will see this exit distribution is not typical, nor is the distribution approximately Binomial or Gaussian on closer inspection. One might note from inspection of the paths that this simulation is not well approximated by a sequence of independent left or right decisions, for example, some shot fall through several levels of pins without striking any, while other shot paths hop horizontally between pins at the same level.

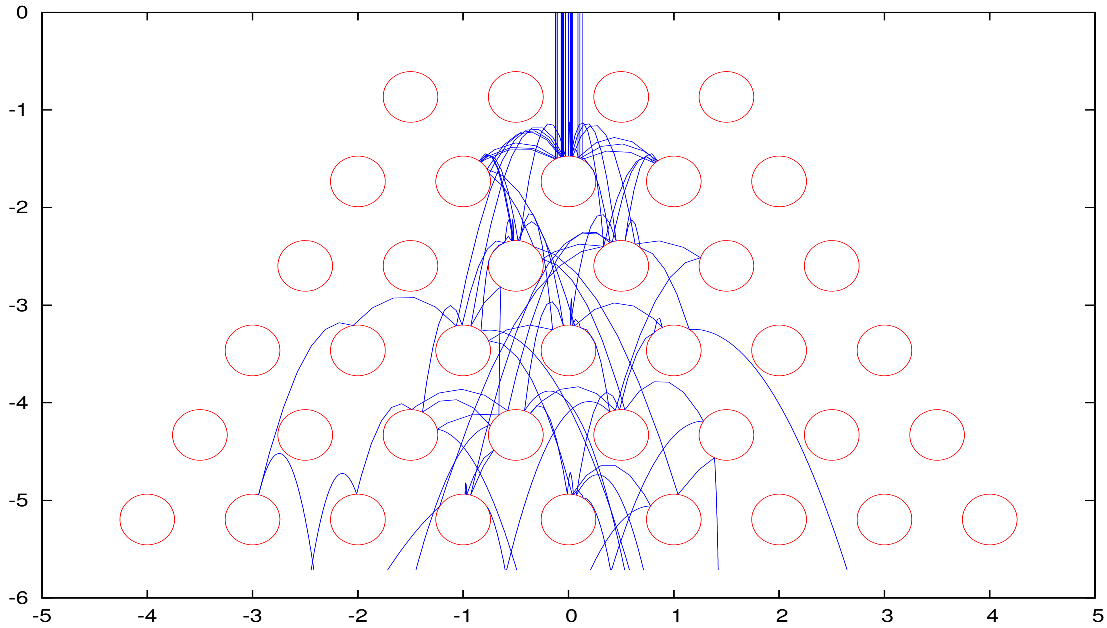

Figure 2: Fifteen simulated shot paths of a quincunx device with inter-pin spacing of $H = 1$cm and $V = \sqrt{3}/2$cm, ball-pin radius of $R = 0.26$cm, and coefficient of restitution of $\gamma = 0.49$. The circles represent the combined radius of ball and pin, the lines are the paths of the ball's centre of mass.

3 The quincunx map

To analyze the quincunx we do not need to consider a complete simulation of a harrow like figure 2; the quincunx can be folded into a compact dynamical system.

Consider five pins arranged in a quincunx cross pattern, with the pins centred at $(-H/2, 0)$, $(H/2, 0)$, $(0, -V)$, $(-H/2, 2V)$, $(-H/2, 2V)$, and a rectangular box with corners at the centres of the four outer pins; see figure 3. The idea is to follow a shot through this box and if the shot exits the box, then it is repositioned on the opposite boundary. This requires only a trivial modification of the algorithm of the last section. Simply compute the times when $x(t) = H/2$, $x(t) = -H/2$, $y(t) = 0$ and $y(t) = -2V$, and if any of these times occur before an impact with any pin, then the shot exits the box and $x(t)$ or $y(t)$ is reset appropriately.

We can now define a three-dimensional discrete-time dynamical system as follows. The state is (x, u, v), where $|x| < H/2 - R$ and $v < 0$, which is taken to be the initial state position and velocity of a shot on the top boundary of the box ($y = 0$). Follow the shot, repositioning when necessary, until it exits through the bottom of the box. The new state is taken from the position and velocity at its exit. One needs to take care that the shot may exit upward through the top of the box; in the following we reposition but do not consider the next exit from the bottom as a state change, that is, a state change only occurs when the shot descends to the a lower level.

We call the three-dimensional discrete-time dynamical system defined as above the *quincunx map*.

4 Experiments

The quincunx map allows quick and compact analysis of a quincunx device. It can be verified that for certain parameters values (for example, $\gamma = 0.49$ and $R = 0.28$cm) the quincunx map has a globally attracting stable orbit of period one, which has a shot path that passes through the side wall of the box once for each iteration of the map, similar to that shown in figure 3(a). In such a quincunx device every shot would eventually be moving either to the left or right at a constant rate.

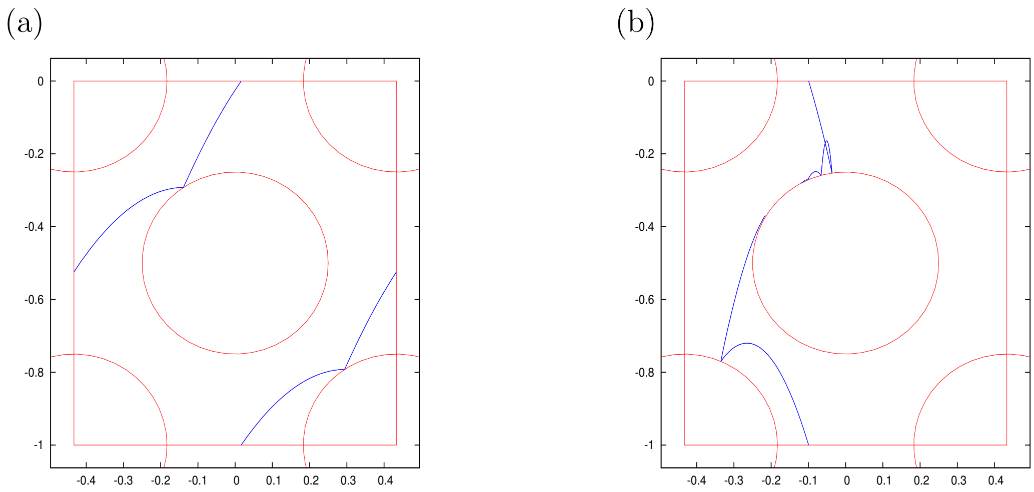

Figure 3: The quincunx map is defined on a box containing a quincunx of pins. Here are shown two examples of periodic orbits that occur for $\gamma = 0.5$ and $R = 0.26$; both are unstable. In (a) the periodic orbit passes through the side of the box and in (b) the periodic orbit that involves a stick-and-roll motion. According to the symbol dynamics we introduce (a) is a type-1 orbit, while (b) is a type 0 orbit.

In general the quincunx map has complex dynamics. There are often many varieties of unstable periodic orbits (see figure 3), which can display multiple impacts and ricocheting between pins, stick-and-roll motions, and vertical motion with, or without, passing through the side walls. Cycle-expansion theory (Cvitanovic, 1988) implies that the behaviour of dynamical systems are strongly influenced by the arrangement of unstable periodic orbits and their Lyapunov spectrum. We have not investigated such properties, for our modest purposes of considering Galton's assumptions there is a simpler method.

Symbolic dynamics (Lind and Marcus, 2000) is a powerful tool for analysing dynamical systems that can be usefully employed here. Consider labelling states of the quincunx map according to the number of times a shot path passes through a side-wall or not, that is, the label is an integer: 0 if the shot path does not pass through a side wall, otherwise the sum of the integers, -1 for each time the shot path passes through the left wall, and $+1$ for each time the shot path passes through the right wall. Any trajectory of the quincunx map defines a symbol sequence, although the method we use here does not uniquely define a path as it would with a generating partition.

The important usage of the symbol sequence for our purposes is to note that if the assumption that shot paths involve independent binomial left or right motions, then half of the symbols in any sequence are expected to be zeros.

Figure 4 shows the computed fraction of zeros in symbols sequences for various R and γ values. When the fraction is 0 or 1 the quincunx map has a stable periodic orbit similar to those shown in figure 3. From figure 4 we see these periodic behaviours punctuate parameters ranges where the fraction of zeros is non-integer, which may correspond to longer more complex periodic orbits or chaotic behaviours. This is typical of nonlinear deterministic systems that display chaos.

Under an assumption of "independent accidents" the fraction of zeros should be around 0.5. A fraction less than this implies a distribution that is broader than Binomial or Gaussian (platykurtic), a higher fraction than 0.5 implies a peaked distribution (leptokurtic). Pho-

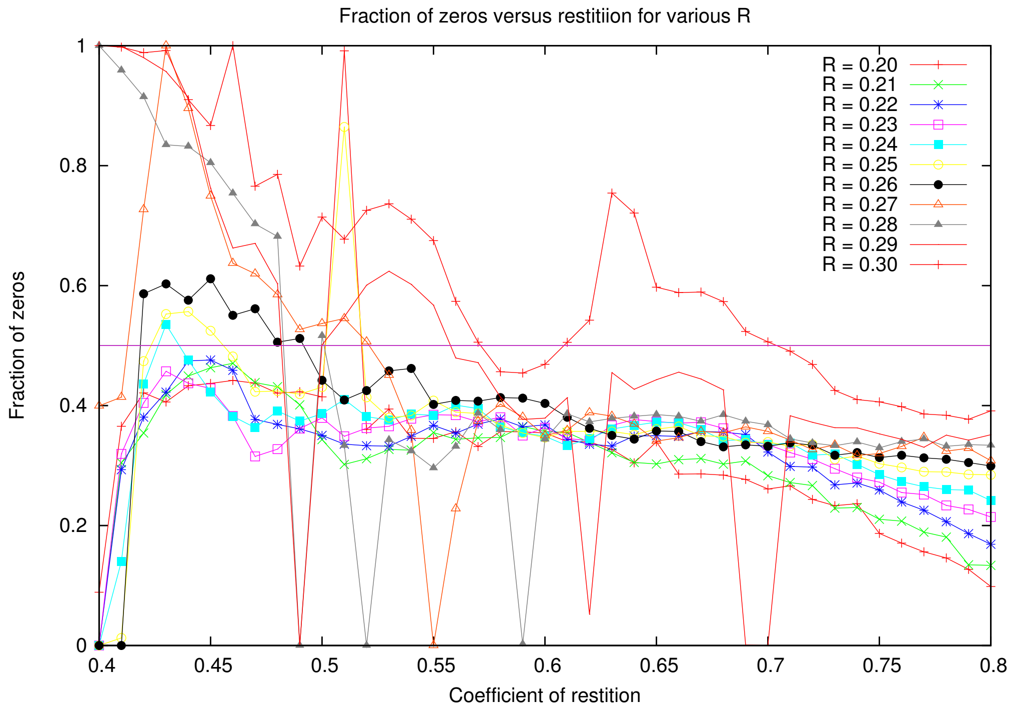

Figure 4: Fraction of zeros in symbolic sequences versus coefficient of restitution γ for various ball-pin radii R. In each case calculated from a sequence of 10000 symbols, where the initial state was $(x, u, v) = (R/2, 0, 0)$ with the first 1000 symbols ignored. The horizontal line at 0.5 indicates the expected fraction of zeros under the "independent accident" assumption.

tographs of Galton's quincunx, and similar modern devices, appear to obtain platykurtic distributions of shot, which figure 4 suggests is the more common behaviour of the quincunx in the parameter ranges we have investigated, which we believe are closest to that of physical devices.

Given that for figure 4 10000 symbols were used to compute the fraction, then in almost all cases the null hypothesis of "independent accidents" is rejected. Even in those cases where the null hypothesis is not rejected we find by using the technique of *context trees* (Kennel and Mees, 2002; Hirata and Mees, 2003) that the symbol sequences have higher order Markov structure. For example, when $R = 0.26$cm and $\gamma = 0.49$, we find the symbol sequence contains only -1, 0 and 1, and transition frequencies given in figure 5(a), which should be compared with the expected transition probabilities of "independent accidents" given in figure 5(b). Applying a goodness-of-fit test, the statistic $\sum_{i,j}(n_{i,j} - Np_{i,j})^2/(Np_{i,j})$, should have a χ^2-distribution with 7 degrees of freedom. The calculated value of this statistic for $N = 1999$ symbols was 36.038, which is significantly greater than the 1% point at 18.4753.

(a)

	-1	0	1	
-1	0.0715	0.1271	0.0470	0.2456
0	0.1325	0.2706	0.1166	0.5197
1	0.0415	0.1221	0.0710	0.2346
	0.2456	0.5197	0.2346	1.0000

(b)

	-1	0	1	
-1	0.0625	0.1250	0.0625	0.25
0	0.1250	0.2500	0.1250	0.50
1	0.0625	0.1250	0.0625	0.25
	0.25	0.50	0.25	1.00

Figure 5: A comparison of (a) observed transition frequencies for $R = 0.26$ and $\gamma = 0.49$ and (b) expected transition probabilities of "independent accidents". Even though the observed fraction of zeros is consistent with "independent accidents", there is significant higher order structure according to a goodness-of-fit test.

5 Conclusions

We have considered the question of whether Galton's quincunx device behaves the way Galton envisaged, and many modern statisticians seem to assume. That is, we have examined whether the assumption that successive impacts of the falling shot are well approximated as a random walk, with "independent accidents". We examine this by considering numerical simulation of a model, which is simplified and certain parameters are unknown. Simulations of the model show that no where in the parameter space of reasonably "realistic" parameters do we find behaviour consistent with the assumptions of a statistical model.

Why the quincunx does not conform to a statistical (random walk) model should be fairly clear: the quincunx map is just a three dimensional deterministic system. That the quincunx map should display significant departures from statistical models should be no surprise. It should be expected to have stable and unstable periodic orbits, intermittency, bifurcations, and other similar features common to low dimensional deterministic systems.

The author believes that conclusions would not change if a more sophisticated model were employed, for example, by taking account of rotational motions, skidding, retardation from impacts with vertical side walls, and so on. The quincunx device is essentially a low dimensional system; its complexity is chaotic, rather than stochastic, and hence better modelled as a deterministic dynamical system.

We can extend our conclusion to a broader perspective. There is no doubt that a statistical model provides a valuable conceptual model of the Galton board, but nonlinear dynamics provides a better model. If one's goal is to make predictions and forecasts, then the dynamical model is surely more informative and accurate, and a statistical model may even be misleading. Consequently, whenever aiming to forecast or predict nonlinear systems one needs to consider carefully the role of nonlinear dynamics and not immediately adopt a statistical framework. For example, in state estimation and forecasting there is significant interest in Kalman filters and various Bayesian and other statistical techniques, but our continuing work indicates that where dynamics plays a significant role, then there is advantage in exploit it (Judd, 2003a,b; Judd et al., 2004; Judd and Smith, 2001, 2004).

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