SOLUTIONS MANUAL

TO

CALCULUS

AN INTUITIVE AND PHYSICAL APPROACH

2ND ED

BY

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Introduction

1. The Solutions In This Manual.

The solutions of all the exercises in the text are given in full. The primary reason is to save professors' time. Choosing exercises for homework assignments can be a laborious matter if one must solve fifteen, twenty or more to determine which are most suitable for his class. A glance at the solutions will expedite the choices.

The second reason is that in many institutions calculus is taught by teaching assistants who have yet to acquire both the training and experience in handling many of the mathematical and physical problems. The availability of the solutions should help these teachers.

2. Suggestions For The Use Of The Text.

The one-volume format of this second edition should give professors more latitude in the choice of topics which might be suitable to the interests of the students or to the length of the course.

Several types of choices might be noted. Because precalculus courses have become more common since the publication of the first edition, some of the analytic geometry topics may no longer have to be taught in the calculus course. The most elementary topics of analytics have been put in an appendix to Chapter 3, Section 4 of Chapter 4, Section 5 of Chapter 7, and the Appendix to Chapter 7. If familiar to the students, all or some can be omitted.

Though I believe strongly in the importance of physical and, more generally, real applications to supply motivation and meaning to the calculus, again class interests and available time must enter into determining how many of these applications can be taken up. I have therefore starred all those sections and chapters which can be omitted without disrupting the continuity.

The last chapter, which is intended as an introduction to the theory or rigor, can be taken up at almost any point after Chapter 10. However, I personally believe that the intuitive approach should be maintained throughout and that this chapter should be left for the last and then taken up only if time permits.

The complete text is intended for a three semester, three hours a week course. However, in view of the number of sections and chapters that are not essential to the continuity the text can be used for shorter courses including those offered in the fourth high school year.

3. Some Additional Topics.

Some physical applications which were included in the first edition were omitted in the second one and replaced in the text proper by applications to economics and to other social science areas. A few of those omitted are reproduced here. They may be useful as suggestions for additional work which bright or somewhat advanced students can undertake, as fill-ins for periods which for one reason or another cannot be used for regular work, or as material for a mathematics club talk. Exercises and solutions relevant to these additional topics are also included here.

A. The Hanging Chain.

In the text proper we derived the equation of the chain or cable suspended from two points (Chap. 16, Sect. 4) on the assumption that the weight per unit length of the cable is the same all along the cable. However, the theory developed there can be used to solve more general problems. One is to determine the shape of the cable if the weight per unit length or, one can say, the density per unit length is specified. The second is, given the desired shape of the cable, how can we fix the distribution of the mass along the cable so that it assumes the desired shape? Both of these problems are readily solved with the theory at hand.

The derivation of (21), the equation of the cable, in the text proper, presupposed that the weight of the cable per unit foot is constant all along the cable. Let us now see what we can do when we let the weight of the cable vary from point to point. Let us denote by $w(s)$ the function that gives the weight per unit foot at point s. Then (11) and (13) still hold, but (14) must be changed to read

$$T_y = \int w(s)\,dx + D. \tag{1}$$

If we divide this equation by (11) and use the fact that T_y/T_x is y', we obtain

$$y' = \frac{1}{T_0}\int w(s)\,ds + D' \tag{2}$$

where D' is D/T_0. If the function $w(s)$ is given, we can calculate $\int w(s)\,ds$. The quantity D' can now be fixed by letting s be 0 at $y' = 0$. We now have y' as a function of s. Next we may proceed as we did in the case where $w(s)$ is a constant and seek to obtain s as a function of x through

$$\frac{ds}{dx} = \sqrt{1 + y'^2}$$

but y' is now given by (2). If the integration can be performed and s is obtained as a function of x, we can substitute this value of s in (2) and attempt to obtain y as a function of x.

We can also solve the second problem. Suppose that we wish to distribute weight along the cable so that the cable hangs in a given shape; that is, we presume that we know the equation of the cable and we wish to find $w(s)$. To solve this problem, we differentiate (2) with respect to x. On the left side differentiation with respect to x produces y''. On the right side to differentiate with respect to x we use the chain rule and differentiate with respect to s and multiply by ds/dx. The derivative of $\int w(s)\,ds$ with respect to s must be $w(s)$ because the integral is that function whose derivative is $w(s)$. Thus our result is

(3) $$y'' = \frac{1}{T_0} w(s) \frac{ds}{dx}.$$

Because we presume that we know the equation of the curve, we can calculate y'' and ds/dx. Hence we can find $w(s)$, that is, the variation of weight along the curve that produces the particular shape of the hanging cable. Of course, the shape of the cable need no longer be a catenary. It is often called a non-uniform catenary.

The theory presented in this section is useful under more general conditions than those so far described. In the derivations of the text and of (2), we attributed the weight to the cable. However, the weight $w(s)$ might be the load on the cable, that is, the load of the bridge itself, if the cable's weight is negligible, or the combined weight of cable and load. In the case of the theory in the text this load would have to be proportional to the arc length of the cable; that is, the load would have to be the same for each unit of length of the cable. In the case of (2), the load could vary along the cable or the combined weight of load and cable could vary along the cable, and the function $w(s)$ would have to represent the variation of the total weight with arc length.

Exercises:

1. Find the law of variation of the mass of a string suspended from two points at the same le vel and acted upon by gravity so that it hangs in the form of a semicircle. Suggestion: Take the semicircle to be the lower half of $x^2+y^2 = 2ay$ and use (3).
2. The derivation given in (2) for a cable whose load varies with arc length applies also to a cable whose load varies with horizontal distance from, say, the lowest point. Thus $T_x = T_o$ and (1) becomes $T_y = \int w(x)dx+D$. Then (2) is $y' = (1/T_o)\int w(x)+D'$. Given that the load per horizontal foot is $w(x) = ax^2+b$, find the equation of the cable.

   Ans. $y = (ax^4+6bx^2)/12T_o$.
3. A heavy chain is suspended at its two extremities and forms an arc of the parabola $y = x^2/4p$. Show that the weight per horizontal foot is constant. Suggestion: Use (3).

Solutions:

1. The lower half of the semicircle is given by $y = a-\sqrt{a^2-x^2}$. Then $y' = x(a^2-x^2)^{-1/2}$ and $y'' = a^2(a^2-x^2)^{-3/2}$, $ds/dx = \sqrt{1+y'^2} = a(a^2-x^2)^{-1/2}$. Then from (3), $w(s) = aT_o/(a^2-x^2)$.
2. Carry out the obvious integrations and use the facts that y' and y are 0 at $x = 0$.
3. We can think of $w(s)ds/dx$ as a function $w(x)$ of x since s is. Now use (3). Since $y = x^2/4p$, $y'' = \frac{1}{2}p$, and $w(x)$ is a constant.

B. Projectile Motion in a Resisting Medium.

After taking up projectile motion in a vacuum (Chap. 18, Sect. 4) one can take up the case of motion in a resisting medium. Since the analysis of projectile motion breaks down into a separate consideration of the horizontal and vertical motions, the mathematics involved, apart from the use of parametric equations, is no more than what was taken up in Chapter 12, Sections 6 and 7, where horizontal and vertical motion in a resisting medium were considered independently. However, the combination of the two and the implications for projectile motion are new and provide an interesting comparison with projectile motion in a vacuum.

The effect of air resistance on the motion of projectiles was first investigated seriously by Newton, Huygens, and Euler. We shall suppose that as the projectile travels through the air, the resistance of the air is proportional to the velocity and directed opposed to that velocity. As in the text, we shall apply Galileo's principle and consider the horizontal and vertical motions separately. Now, the horizontal and vertical components of the velocity at any point of the projectile's path are $\dot{x}$ and $\dot{y}$, respectively, where x and y are functions of t that represent the horizontal and vertical motion and the dot means differentiation with respect to t. Hence the air resistance, because it is oppositely directed, should have the components $-K\dot{x}$ and $-K\dot{y}$, where K is the proportionality constant.

To obtain the parametric equations of the motion, we use Newton's second law of motion, namely, that the net force acting must equal the mass times the acceleration of the projectile. We shall apply this law to the horizontal and vertical motions separately. Suppose that the projectile is shot out with an initial velocity of magnitude V inclined at angle A to the ground. This initial velocity does give the projectile a constant horizontal velocity of $V \cos A$ but no acceleration in the horizontal direction and therefore no continuously acting force in the horizontal direction. However, there is a horizontal force acting at any time t, namely the air resistance $-K\dot{x}$. Hence Newton's second law says for the horizontal motion that $ma = -K\dot{x}$. However, the horizontal acceleration is the time derivative of the horizontal velocity so that

$$m\ddot{x} = -K\dot{x}.$$

If we divide both sides by m and replace K/m by k, we obtain

$$\ddot{x} = -k\dot{x}. \tag{1}$$

In the case of the vertical motion there are two forces acting at any time t, the force of gravity which is $-32m$ and the vertical component, $-K\dot{y}$, of the air resistance (the upward direction is positive). Hence the differential equation for the vertical motion is, by Newton's second law,

$$m\ddot{y} = -32m - K\dot{y}.$$

If we divide both sides of this equation by m and again replace K/m by k, we obtain

$$\ddot{y} = -32 - k\dot{y}. \tag{2}$$

To integrate (1) we first write it in a more familiar form. Since $\dot{x} = v_x$, we have

$$\frac{dv_x}{dt} = -kv_x .$$

The right-hand side contains the dependent variable, so we convert to

$$\frac{dt}{dv_x} = \frac{-1}{kv_x}.$$

Then

$$t = -\frac{1}{k}\log v_x + C$$

or, by solving for v_x,

$$v_x = e^{-kt-C} = e^{-kt}e^{-C} = De^{-kt},$$

where $D = e^{-C}$. Because $v_x = V\cos A$ when $t = 0$,

(3) $$v_x = V\cos A e^{-kt}.$$

Equation (3) should be compared with (6) of the text proper. Now $v_x = dx/dt$; hence, since $x = 0$ when $t = 0$,

(4) $$x = \frac{V\cos A}{k}(1 - e^{-kt}).$$

This equation should be compared with (7) of the text. In (3) and (4) we have the formulas for the horizontal velocity and horizontal distance traveled as functions of t.

To integrate (2) we perform similar steps. We write $\dot{y}$ as v_y. Then (2) reads

$$\frac{dt}{dv_y} = -\frac{1}{32+kv_y}.$$

By integration

$$t = -\frac{1}{k}\log(32+kv_y) + C$$

or

$$32 + kv_y = e^{-kt-kC} = De^{-kt},$$

where $D = e^{-kC}$. Then

(5) $$v_y = \frac{1}{k}(De^{-kt}-32).$$

We know that $v_y = V\sin A$ when $t = 0$. Substitution of these values yields

$$D = kV\sin A + 32,$$

and so (5) becomes

(6) $$v_y = V\sin A e^{-kt} + \frac{32}{k}e^{-kt} - \frac{32}{k}.$$

By one more integration we obtain

$$y = -\frac{V \sin A}{k} e^{-kt} - \frac{32t}{k} + C.$$

Because y = 0 when t = 0,

$$0 = -\frac{V \sin A}{k} - \frac{32}{k^2} + C.$$

We solve for C and substitute its value in the preceding equation. Then

$$y = -\frac{V \sin A}{k} e^{-kt} - \frac{32}{k^2} e^{-kt} - \frac{32t}{k} + \frac{V \sin A}{k} + \frac{32}{k^2}$$

or

$$y = -\frac{32t}{k} + \left(\frac{V \sin A}{k} + \frac{32}{k^2}\right)(1 - e^{-kt}). \tag{7}$$

We have in (6) and (7) the formulas for the vertical velocity and height above ground of the projectile. These formulas should be compared with (8) and (9) respectively of the text.

It would, of course, be interesting to determine what effect air resistance has by comparing results obtained here with the results obtained for projectile motion without air resistance. A few of these comparisons will be left for the exercises. (See also the work on infinite series in Chap. 20, Sect. 12.) At the moment we might mention that the projectile fired with a fixed initial velocity V, at a fixed angle A to the ground, and moving in the resisting medium as opposed to a vacuum takes less time to reach maximum height, reaches maximum height closer

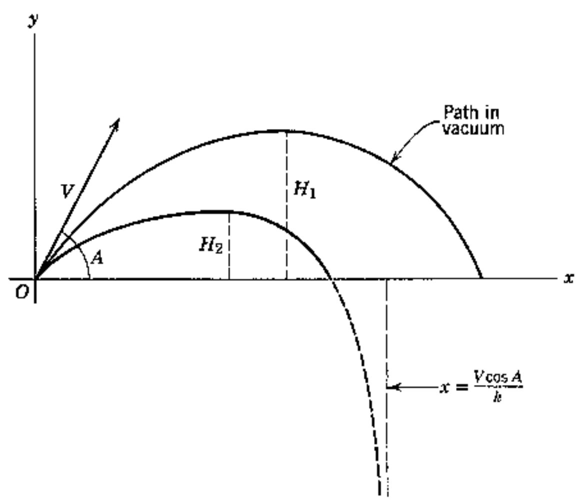

Figure 1.

to the gun (Fig. 1), and attains less maximum height than if fired in a vacuum. Moreover, the maximum height H_2 is attained beyond the midpoint of the range. The first part of the path is straighter than in a vacuum and the latter part steeper. The projectile strikes the ground at a steeper angle and with less speed than that with which it was fired. Finally, the maximum range is obtained at an angle of fire of less than 45°.

Exercises

1. Find the direct equation relating y and x by eliminating t between (4) and (7).

$$\underline{\text{Ans.}}\ y = \left(V \sin A + \frac{32}{k}\right)\frac{x}{V \cos A} + \frac{32}{k^2}\log\left(1 - \frac{kx}{V \cos A}\right).$$

2. Find the time t_2 it takes the projectile to reach the highest point of its path.

$$\underline{\text{Ans.}}\ t_2 = \frac{1}{k}\log\left(1 + \frac{kV \sin A}{32}\right).$$

3. Find the coordinates (x_2, y_2) of the highest point of the projectile's path.

$$\underline{\text{Ans.}}\ x_2 = \frac{V^2 \sin A \cos A}{32 + kV \sin A},\ y_2 = \frac{V \sin A}{k} - \frac{32}{k^2}\log\left(1 + \frac{kV \sin A}{32}\right).$$

4. Show that the projectile moving in the resisting medium attains its maximum height at a value of x closer to the starting point than it does when shot out at the same angle A and with the same initial velocity V in a vacuum. Suggestion: Compare (15) of the text and the value of x_2 in Exercise 3.
5. As a check on the results in Exercises 1 and 3, we could do the following. We know that the slope of the projectile's path is 0 at the maximum height. Calculate dy/dx from the result in Exercise 1, substitute in it the value of x_2 given in Exercise 3, and see if the slope is 0.
6. (a) What is the terminal horizontal velocity, that is, the velocity as t becomes infinite, of the projectile motion discussed in the text? Ans. 0.
   (b) What is the terminal vertical velocity? Ans. $-32/k$.
   (c) Using the results of parts (a) and (b), describe the path as t becomes infinite.
7. Using the result of Exercise 1, describe the path as x approaches the value $(V \cos A)/k$. Does the result agree with the answer to part (c) of Exercise 6?
8. An airplane flying horizontally with speed U releases a bomb of mass m. If the air resistance is km times the velocity, where k is a proportionality constant, show that the horizontal and vertical distances traveled by the bomb in time t are $x = U(1 - e^{-kt})/k$ and $y = 32(e^{-kt} - 1 + kt)/k^2$, respectively.
9. A bomb is released from an airplane traveling horizontally at a speed of U ft/sec and at an altitude of H feet. If the air resistance of the bomb is km times the velocity, where k is a proportionality constant, show that the path of the bomb t seconds after its release will be inclined to the horizontal at the angle $\tan^{-1}[32(e^{kt} - 1)/kU]$.

Solutions to the Exercises on Projectile Motion

1. From (4) we have $1 - e^{-kt} = kx/V\cos A$ and $t = (-1/k)\log(1 - kx/V\cos A)$. Inserting these values in (7) gives the text's answer.
2. At the highest point $\dot{y}$ or v_y is 0. Hence use (6) to solve for t when $v_y = 0$.
3. Use the value of t_2 obtained in Exercise 2 and substitute for t in (4) and (7). We have from Exercise 2 that $e^{kt} = (32 + kV\sin a)/32$ and e^{-kt} is the reciprocal. Hence $1 - e^{-kt} = kV\sin A/(32 + kV\sin A)$. The rest is straightforward to get the text's results.
4. Following the suggestion we must show that $V^2\sin A\cos A/(32 + kV\sin A)$ is less than $V^2\sin 2A/64$. Rewrite the first of these quantities as $V^2\sin 2A/(64 + 2kV\sin A)$. Since $2kV\sin A$ is positive, the expression is less than $V^2\sin 2A/64$.
5. Calculate dy/dx from the result in Exercise 1 and substitute for x the value of x_2 given in Exercise 3. Mere algebra shows $dy/dx = 0$.
6 (a) As t becomes infinite, e^{-kt} approaches 0. Then, by (3), v_x approaches 0.
   (b) By (6), v_y approaches $-32/k$.
   (c) Since the horizontal velocity approaches 0 and the vertical velocity approaches a constant the path must approach more and more a vertical straight line.
7. As x approaches $V(\cos A)/k$, y in Exercise 1 approaches $-\infty$. This agrees with 6(c).
8. We start with (1) as in the text here. However, when $t = 0$, $v_x = U$. Hence in our case $v_x = Ue^{-kt}$. Then, integrating, and using $x = 0$ when $t = 0$ (which means x is measured from the point where the bomb is released, we get $x = U(1 - e^{-kt})/k$. If we measure distance downward as positive we have for the vertical motion as in (2) (except for sign) $\ddot{y} = +32 - k\dot{y}$ and so by using the method of the text here in the derivation of (5) we obtain $\dot{y} = (32 - De^{-kt})/k$. Since $\dot{y} = 0$ when $t = 0$, $D = 32$ and $\dot{y} = (32 - 32e^{-kt})/k$. Integrating and using $y = 0$ when $t = 0$, we get the result for y.
9. We may take over from Exercise 8 that $v_x = Ue^{-kt}$ and $v_y = (1/k)(32 - 32e^{-kt})$. Then the direction of the bomb is given by $\tan\theta = v_y/v_x = (32 - 32e^{-kt})/kUe^{-kt}$.

C. The Brachistochrone Problem.

This problem, one of the famous ones in the history of mathematics, like the preceding topic, belongs under the subject of rectangular parametric equations (Chap. 18).

In Section 8, the text proper takes up tangential and normal acceleration along curves and arrives at the formula (78):

$$\dot{s}^2 = 64(y_0 - y). \tag{1}$$

This says that the velocity acquired by an object which slides along a curve under the action of gravity and starts at the point (x_o, y_o) with zero velocity is dependent only on the vertical distance fallen and is independent of the shape of the curve. (The arc length s is measured from (x_o, y_o)).

With this result at our disposal we can examine the proof that John Bernoulli (1667-1748) gave in 1697 of the brachistochrone property of the cycloid. The term brachistochrone means shortest time and it enters in the following way. Suppose that a particle starts from rest and is allowed to slide along a curve (Fig. 2) from O to B under the action of gravity. What should the shape of the curve be in order that the time of travel be least?

One's first thought is that the curve joining O and B should be a straight line. This curve would indeed furnish the shortest distance from O to B, but it need not be the one that makes the time of travel least. If a curve is used that is steeper at O than the line OB, the

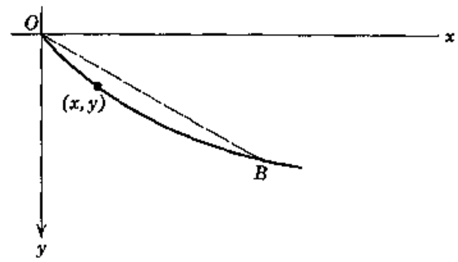

Figure 2

tangential acceleration caused by gravity will be greater, and therefore the velocity acquired will be greater at least at the outset. If the incipient velocity is greater and because the particle gains velocity as it travels along the curve, it may still take less time to traverse a curved path from O to B even though this curved path is longer than the straight line path from O to B.

Let us consider, then, a curve from O to B. We choose the coordinate axes so that y is positive downward. We learned in (1) that the velocity along the curve at a point (x,y) is the vertical distance traveled by the particle if it starts from rest. Because our particle starts from rest at O and y = 0 at O, then by (1)

(2) $$\frac{ds}{dt} = v = 8\sqrt{y}\ .$$

This equation is correct, but it certainly does not incorporate any condition about the curve being the one for which the time of travel is least. It is, in fact, true for any curve.

Here John Bernoulli applied a brilliant thought. He said, suppose that light were to travel from O to B with a variable velocity v given by (2). According to Fermat's principle, light always takes the least time. Perhaps if we analyzed how light travels when the velocity varies, we might obtain the clue to the solution of our problem. Now light changes speed when it passes from one medium into another. Indeed, we showed (formula (16) of Chapter 8) that when light passes from a medium in which its velocity is v_1 to another in which its velocity is v_2, then

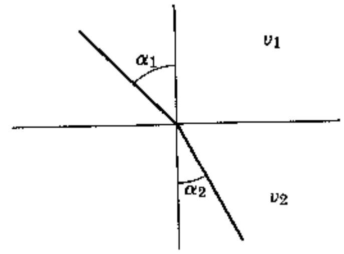

Figure 3

$$(3) \qquad \frac{v_1}{v_2} = \frac{\sin \alpha_1}{\sin \alpha_2}$$

where α_1 and α_2 are the angles of incidence and refraction and are the angles shown in Fig. 3. This is Snell's law of refraction of light. Note that if $v_2 > v_2$, $\alpha_2 > \alpha_1$.

The law of refraction applies when there is a sudden or discontinuous change in the velocity of light. However, Bernoulli wished to consider the behavior of light when it travels with a continuously changing velocity. He therefore supposed that the space from O to B was broken up into a series of layers (Fig. 4) within each of which the velocity is constant. Suppose now that light passes from the i^{th} layer in which the velocity is v, to the (i+1)-st layer in which the velocity .

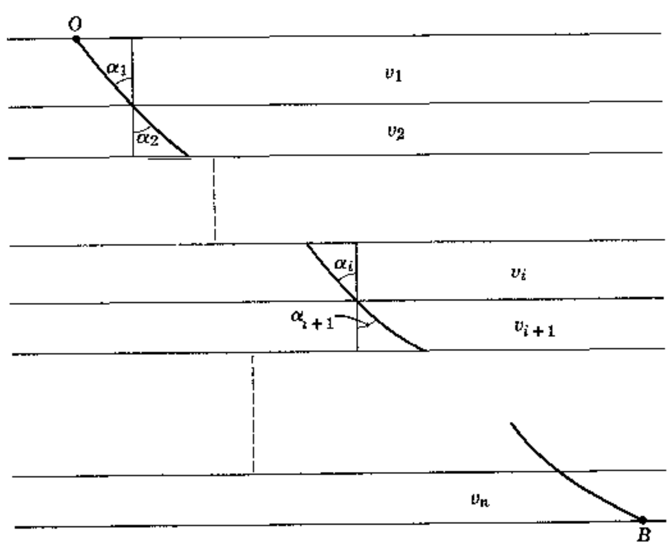

Figure 4

is v_{i+1}. Let α_i be the angle of incidence and α_{i+1} the angle of refraction. Then according to (3)

$$\frac{\sin \alpha_i}{v_i} = \frac{\sin \alpha_{i+1}}{v_{i+1}} .$$

This equation holds at the boundary of each layer, so that

$$\frac{\sin \alpha_1}{v_1} = \frac{\sin \alpha_2}{v_2} = \cdots = \frac{\sin \alpha_n}{v_n} .$$

what this equation says is that

$$\frac{\sin \alpha_i}{v_i} = \text{constant.} \tag{4}$$

Let us now increase the number of layers between the level of O and the level of B. Then (4) will hold at each boundary. If the number of layers becomes infinite, each horizontal line between the horizontal through O and the horizontal through B becomes a boundary, and we have in place of (4) that

$$\frac{\sin \alpha}{v} = \text{constant} \tag{5}$$

where α and v are now functions of y. Moreover, if we think of light as following the curved path OB (Fig. 5), the direction α of the incident and refracted light at any point (x,y) is the angle that the tangent to

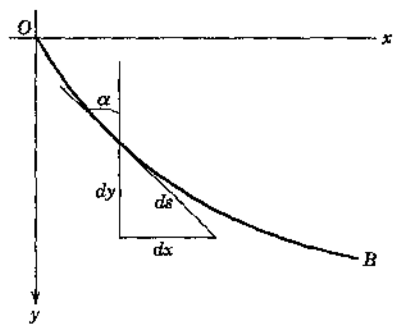

Figure 5

the curve at (x,y) makes with the vertical. However, we also see from Fig. 5 that

$$\sin \alpha = \frac{dx}{ds}$$

so that (5) becomes

$$\frac{\frac{dx}{ds}}{v} = \text{constant}$$

or, because dx/ds = 1/(ds/dx),

$$v = \frac{1}{B\frac{ds}{dx}} = \frac{C}{\sqrt{1 + (\frac{dy}{dx})^2}}$$

where B is some constant and C = 1/B.

Thus through the study of light we have learned something about v. We now use this fact in (2) and write

$$\frac{C}{\sqrt{1 + (\frac{dy}{dx})^2}} = 8\sqrt{y} \ .$$

If we solve for dy/dx, we obtain

(6) $$\frac{dy}{dx} = \sqrt{\frac{D - y}{y}}$$

where D is a new constant.

We now try to integrate (6). Because the right side is a function of y, let us invert and write

$$\frac{dx}{dy} = \sqrt{\frac{y}{D - y}}$$

so that

$$x = \int \frac{\sqrt{y}}{\sqrt{D - y}}\, dx$$

In view of the presence of the radical, let us use the change of variable

(7) $$y = D \sin^2 u.$$

Then

$$x = \int \frac{\sin u}{\cos u}\, 2D \sin u \cos u \, du$$

or

$$x = \int 2D \sin^2 u \, du.$$

The integral is readily evaluated by changing $\sin^2 u$ to $(1 - \cos 2u)/2$ so that

$$x = \frac{D}{2}(2u - \sin 2u) + C.$$

Now, $x = 0$ when $y = 0$ and when $y = 0$, $u = 0$. Then $C = 0$ and

$$x = \frac{D}{2}(2u - \sin 2u)\ . \tag{8}$$

We could now solve (7) for u and put this value of u in (8) to obtain the equation relating x and y. However, we can equally well take the equations

$$x = \frac{D}{2}(2u - \sin 2u)\ ,$$

$$y = D\sin^2 u = \frac{D}{2}(1 - \cos 2u)$$

as the parametric equations of the curve with u as the parameter. We can let $2u = \theta$ and let $D/2 = R$ so that the equations become

$$\begin{aligned} x &= R(\theta - \sin\theta) \\ y &= R(1 - \cos\theta)\ , \end{aligned} \tag{9}$$

and we can now see that the curve is a cycloid.

We do want the curve to pass through the given point B. If the coordinates of B are (x_1,y_1), we want the cycloid which for some value of θ and some value of R yields (x_1,y_1). That is, we must have

$$x_1 = R(\theta_1 - \sin\theta_1),$$

$$y_1 = R(1 - \cos\theta_1)\ .$$

These equations do determine R and do it so that for this value of R the equations (9) will pass through (x_1,y_1) when $\theta = \theta_1$.

Thus John Bernoulli showed that the cycloid is the curve along which a particle slides under the action of gravity from one point to another in least time.

This problem is a peculiar one insofar as the calculus is concerned. We note that it is a kind of minimum problem. That is, we did seek something that would make time of travel least. However, we did not, as one does in the usual maxima and minima problems, find a value of x at which some dependent variable y is least. We found a <u>curve</u> for which the dependent variable, time, is least. Such problems cannot usually be done with the calculus and, in fact, must be handled with the techniques of a branch of mathematics called the calculus of variations, which is an extension of the calculus. The solution of the brachistochrone problem by means of the calculus proper was possible only because Bernoulli used an ingenious argument.

<u>Exercises</u>

1. Would you say that Bernoulli used an entirely mathematical argument to solve the brachistochrone problem?

2. Would you say that Bernoulli was able to solve the brachistochrone by relying entirely upon mathematics and concepts of mechanics such as velocity and acceleration?
3. Specifically, what did Bernoulli accomplish by introducing the motion of light?
4. What is the essence of the argument that the cycloid requires least time?

Solutions

1. No. He used the physical fact that light takes the path requiring least time to obtain an important fact about the velocity of motion.
2. No. He used the principle of least time which as Bernoulli used it, is a principle of optics, not mechanics.
3. The key fact obtained by studying the motion of light is that $v = C/\sqrt{1+(dy/dx)^2}$. This tells us how the velocity of the motion must be related to the slope of the curve along which the motion takes place if the time of travel is to be least.
4. There are two key ideas in Bernoulli's proof. The first is that for motion along a curve under the action of gravity (starting from rest) the velocity attained is $v = 8\sqrt{y}$ where y is the vertical distance fallen. However, as the text points out, this fact holds for any curve. The problem is to single out the curve requiring least time. Here Bernoulli calls upon the behavior of light. The principle of least time implies the law of refraction (Chapter 8, formula (16)). The law of refraction extended to a continuous change in the medium implies $v = C/\sqrt{1+(dy/dx)^2}$. This equation relates the slope of the curve requiring least time to the velocity. However, the velocity is not uniquely fixed by this last equation. If we now add that the velocity at any level is $8\sqrt{y}$, that is, the velocity determined by gravity, to the condition which least time imposes, we get enough information to determine the unique curve along which the particle must move.

D. Kepler's Laws.

In the text we arrived at formula (35), namely

$$\rho = \frac{\frac{h^2}{GM}}{1+e\cos(\theta+\alpha)}, \tag{1}$$

as the equation of the path of a planet which is attracted to the sun in accordance with the law of gravitation. We then sought to determine e, primarily, so that we could learn which conic section is the actual path.

To simplify the work we adopted the initial conditions that at time $t = 0$ the planet is on the polar axis at a distance ρ_0 from the pole (which is the location of the sun) and that the planet has at $t = 0$ a velocity v_0 which is perpendicular to the polar axis. These initial conditions enabled us to determine the nature of the conic section and it turns out that whether the conic is an ellipse, parabola or hyperbola depends on the value of v_0.

One can, with the mathematics at our disposal, deduce a more general result which is of value to mathematical astronomers. Instead of supposing that v_o is perpendicular to the polar axis we allow it to make an angle A to that axis. The position of the planet at time $t = 0$ will still be on the polar axis at a distance ρ_o from the sun. Moreover, we do not suppose that the line from focus to directrix is the polar axis so that α need not be 0 or π. Under these more general initial conditions we can still determine e, as well as α and h and we arrive at the surprising conclusion that only the magnitude of v_o but not the direction A which it makes with the polar axis determines the particular conic section.

The derivation of this conclusion under the more general initial conditions is somewhat lengthy but elementary.

Let us suppose that the planet starts out at some point P_o in space (fig. 6) whose distance ρ_o from the sun at O is known. Further at P_o suppose the planet has an initial spped v_o whose direction makes an angle A with the line joining O to P_o. We choose the polar axis of our polar coordinate system to be the line OP_o so that θ is measured from OP_o counterclockwise. What these initial conditions tells us is that

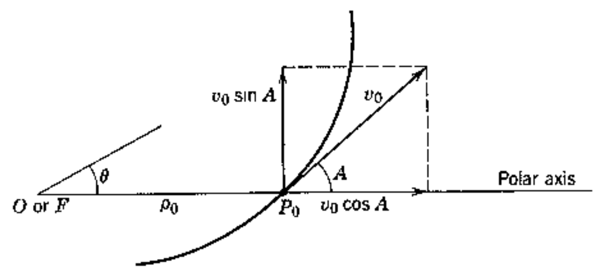

Figure 6

(2) $$\text{at } \theta = 0, \quad \rho = \rho_o .$$

Since $v_o \cos A$ is the radial component of the velocity and v_ρ is $\dot{\rho}$, then

(3) $$\dot{\rho}_o = v_o \cos A$$

where the subscript in $\dot{\rho}_o$ denotes the value of $\dot{\rho}$ at $\theta = 0$. Likewise since $v_o \sin A$ is the transverse component of the velocity and $v_\theta = \rho\dot{\theta}$ then

(4) $$\rho_o \dot{\theta}_o = v_o \sin A .$$

We must understand in (4) that θ_o is 0 but $\dot{\theta}_o$ is the time rate of change of θ at $\theta = 0$ and this is not zero.

We can determine h at once. By (19) of the text proper $h = \rho^2 \dot{\theta}$. Since h is a constant we can use its value at $0 = 0$. Then by (4),

(5) $$h = \rho_o v_o \sin A .$$

We are now going to obtain relations involving α and e which will enable us to determine both. From (1) when $\theta = 0$ and so $\rho = \rho_o$

$$\rho_o = \frac{h^2}{GM(1+e\cos\alpha)}$$

from which we have, by solving for $e\cos\alpha$,

$$e\cos\alpha = \frac{h^2}{GM\rho_o} - 1.$$

Then, in view of (5),

$$e\cos\alpha = \frac{\rho_o v_o^2 \sin^2 A}{GM} - 1. \tag{6}$$

If we could get a value for $e\sin\alpha$ we would be able to use it and (6) to find e and α separately. To involve $\sin\alpha$ we go back to (1) and differentiate. The algebra is simpler if we first write (1) as

$$\frac{1}{\rho} = \frac{GM[1+e\cos(\theta+\alpha)]}{h^2}\ .$$

Now

$$-\frac{1}{\rho^2}\frac{d\rho}{d\theta} = -\frac{GM}{h^2}e\sin(\theta+\alpha)\ . \tag{7}$$

By setting $\theta = 0$ we can get an expression for $e\sin\alpha$ but this expression would do no good because we do not know the value of $\frac{d\rho}{d\theta}$ at $\theta = 0$. However, we have in (3) the value of $\dot{\rho}_o$ or $\frac{d\rho}{dt}$ at $\theta = 0$. This suggests that we use the fact that

$$\frac{d\rho}{d\theta} = \frac{d\rho}{dt}\frac{dt}{d\theta}\ .$$

By (19) of the text proper,

$$\frac{d\rho}{d\theta} = \frac{d\rho}{dt}\frac{\rho^2}{h}\ .$$

Substitution of this result in (7) gives

$$\frac{d\rho}{dt} = \frac{GM}{h}e\sin(\theta+\alpha)\ .$$

and at $\theta = 0$

$$\dot{\rho}_o = \frac{GM}{h}e\sin\alpha\ .$$

Then

$$e\sin\alpha = \frac{h}{GM}\dot{\rho}_o\ ,$$

and in view of (5) and (3)

(8) $$e \sin \alpha = \frac{\rho_o v_o^2 \sin A \cos A}{GM}.$$

With (6) and (8) we can obtain e. We square (6) and (8) and add. Then

(9) $$e^2 = 1 - \frac{2\rho_o v_o^2 \sin^2 A}{GM} + \frac{\rho_o v_o^4 \sin^2 A}{G^2 M^2}.$$

To obtain α we have but to divide (8) by (6). Thus

(10) $$\tan \alpha = \frac{\rho_o v_o^2 \sin A \cos A}{\rho_o v_o^2 \sin^2 A - GM}.$$

Having obtained (9) and (10), let us try to profit from them. We can write (9) as

(11) $$e^2 = 1 + \frac{\rho_o v_o^2 \sin^2 A}{GM} \left(\frac{\rho_o v_o^2}{GM} - 2\right).$$

If $\rho_o v_o^2/GM < 2$, the quantity in parentheses will be negative, and because all other quantities are positive, e^2, which is necessarily positive, will lie between 0 and 1. Then the conic section will be an ellipse. We may put this statement thus:

(12a) if $v_o < \sqrt{\frac{2GM}{\rho_o}}$, the path is an ellipse.

From (11) we also see that if $\rho_o v_o^2/GM - 2 = 0$, then $e = 1$, or

(12b) if $v_o = \sqrt{\frac{2GM}{\rho_o}}$, the path is a parabola.

Finally, if $\rho_o v_o^2/GM > 2$, then $e > 1$, or

(12c) if $v_o > \sqrt{\frac{2GM}{\rho_o}}$, the path is a hyperbola.

We see, then, that only the magnitude of the initial velocity but not the direction determines the particular conic section that the attracted body follows. The shape and location of the particular conic section does depend on angle A but the fact that it is an ellipse, say, does not.

The problem solved in the text proper under the more specialized initial conditions or the same problem with the more general initial conditions treated here is known as the simplified or modified two-body problem. The simplification consists in assuming that the sun is fixed and that a single planet is attracted to the sun. Actually each body, sun and planet, attracts the other and both move. This more general problem and extensions to the three-body and n-body problems are treated in texts on celestial mechanics but the mathematics involves far more of the subject of differential equations than can be taken up in the calculus. Moreover, exact solutions cannot be obtained.

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New York City

Solutions to Chapter 2

CHAPTER 2, SECTION 1

1. Independent variable is n; dependent variable is A.
2. $t = v/32$. Yes.
3. $x = \pm\sqrt{y/5.3}$. No.
4. (a) $A = \pi r^2$; (b) $A = \pi d^2/4$.
5. $r = \pm\sqrt{A/\pi}$; no; the positive root because real radii are positive lengths.
6. $\sqrt{x^2-9}$
7. 30 miles per hour is $30\cdot 5280/60\cdot 60$ or 44 ft/sec. Hence $d = 44t$.
8. Other side is $4/x$. Hence $p = 2x + 2(4/x)$.

CHAPTER 2, SECTION 1, SECOND SET

2. $f(0) = 0$; $f(2) = -22$; $f(-2) = 14$; $f(9) = -162$; $f(-9) = 0$.
3. Replace a in answers given by x_0.
4. (a) $f(x) = x - (1/x)$; $f(-x) = -x - (1/-x) = -x + 1/x$. Hence $f(-x) = -f(x)$.
   (b) $f(1/x) = (1/x) - 1/(1/x) = 1/x - x = -f(x)$.
5. $f(0) = -9/7$; $f(2) = -5/3$; $f(-2) = -5/3$; $f(\sqrt{7})$ has no value.
6. $f(2x) = (2x)^2 - 7(2x) = 4x^2 - 14x$; $f(x+h) = (x+h)^2 - 7(x+h) = x^2 + 2xh + h^2 - 7x - 7h$.
8. $f(-2) = 9$ and $g(-2) = -15$. Hence $f(-2)\cdot g(-2) = -135$.
9. $f(3) = -26$; $f(-1) = -2$; $f(1/x) = 3/(1/x) - (1/x)^3 = 3x - (1/x^3)$.
10. $f(0) = 8$; $f(4) = 6$; $f(g^2) = (g^4 + 32)/(g^2 + 4)$.
11. $g(-x) = (-x)^3 = -x^3 = -g(x)$.
12. $g(-x) = (-x)^4 + 2(-x)^2 + 1 = x^4 + 2x^2 + 1 = g(x)$
13. The fallacy is that we cannot choose $f(\sin x)$ to be $x \sin x$. This is not a function of $\sin x$ alone but of x and $\sin x$.

CHAPTER 2, SECTION 2

1. (a) Parabola; vertex (0, 0); opening upwards.
   (b) Upper half of circle; radius 1; center (0, 0).
   (c) Lower half of circle; radius 1; center (0, 0).
   (d) Hyperbola with asymptotes $x = 3$, $y = 0$. See (3) of the text.
   (g) Straight line through (−3, 0) and (0, 3) with the point (3, 6) omitted.
   (h) Caution: The point (1, 3) is not on the graph.
   (m) The graph consists of two half lines emanating from the origin and extending diagonally upward to the left and to the right. The graph is the same as for $y = |x|$.
2. (a) Graph is the same as for $k = 16 + h$ except that the point (0, 16) is not included.
   (b) Graph is the same as for $k = h^2$ except that (0, 0) is not included.
   (c) Graph is the same as for $k = 9 + h^2$ except that (0, 9) is not included.
   (d) Graph is the same as for $k = 9h + h^2$ except that (0, 0) is not included.
   (e) Graph is the same as for $k = h - 1$ except that (1, 0) is not included.
3. The limit is 3.

CHAPTER 2, SECTION 3, FIRST SET

1. Yes, because there is less and less time for the speed to change prior to the third second.
2. 128 ft/sec.

CHAPTER 2, SECTION 3, SECOND SET

1. Let the position be s_1 at time t_1 and s_2 at time t_2. The change in distance is $s_2 - s_1$; the rate of change of distance is $(s_2 - s_1)/(t_2 - t_1)$.
2. In the notation of Exercise 1, $t_1 = 3$, $t_2 = 5$, $s_1 = 16(3)^2$, $s_2 = 16(5)^2$. Thus distance traveled from $t = 3$ to $t = 5$ is 256 ft; the rate of change of distance or average speed is 128 ft/sec.
3. The average speed is the distance traveled during some interval of time divided by the interval. Instantaneous speed is the speed at an instant of time and is obtained as a limit of average speeds as the interval of time which starts or ends at the instant approaches 0.
4. The limit concept.
5. (a) $t_1 = 0$, $t_2 = 5$; hence $(s_2 - s_1)/(t_2 - t_1) = 80$ ft/sec. (b) $t_1 = 4$, $t_2 = 5$; hence $(s_2 - s_1)/(t_2 - t_1) = 144$ ft/sec. (c) Calculate the average speed for the interval 5 to 6, 5 to 5.1, 5 to 5.01, etc. The limit, determined at this stage only by seeing what number the average speeds seem to be approaching, is 160 ft/sec.
6. The result should be the same as that obtained in the text, namely 128 ft/sec.
7. Not necessarily; the speed may change at any instant or instants during the next hour.

CHAPTER 2, SECTION 4

1. (a) The limit of $3h^2$ as h approaches 0 is 0; the limit of h^2 as h approaches 0 is 0. For the purpose of finding the limit of $3h^2/h^2$ as h approaches 0, we may divide numerator and denominator by h^2. The quotient is 3 and the limit of this quotient as h approaches 0 is 3 because the quotient is always 3.
   (b) Use (a) as the model. The limits are 0, 0, $\frac{1}{2}$.
   (c) As in (a) the limits of numerator and denominator are 0 and 0. To find the limit of the quotient we may divide numerator and denominator by h, obtaining $3h + 1$. The limit as h approaches 0 is 1.
   (d) The method is the same as in (a). The limits of the numerator and the denominator are 0 and 0. The quotient after division of numerator and denominator by h is $3h^2 + 3h + 1$. The limit as h approaches 0 is 1.
   (e) Same method as in (a). The limits are 0, 0, 0.
   (f) Same method as in (a). The limits are 0, 0, 0.

2. Paraphrase what is done in Section 4.
3. (a) In the notation of the text, $s_3 = 16\cdot 3^2 = 144$, $s_3 + k = 16(3 + h)^2$, and $k/h = 96 + 16h$. The limit of $96 + 16h$ as h approaches 0 is 96.
   (b) The same process as in (a); here, however, $s_5 = 16(5)^2$, $s_5 + k = 16(5 + h)^2$, and $k/h = 160 + 16h$. The limit of $160 + 16h$ as h approaches 0 is 160.
   (c) The same process as in (a); here, however, $s_6 = 16(6)^2$, $s_6 + k = 16(6 + h)^2$, and $k/h = 192 + 16h$. The limit as h approaches 0 of $192 + 16h$ is 192.
4. (a) The value of s when $t = 3$ is 240 ft.
   (b) $(s_4 - s_3)/(4 - 3) = (256 - 240)/1 = 16$ ft/sec.
   (c) We go through the method of increments. $s_3 = 240$, $s_3 + k = 128(3 + h) - 16(3 + h)^2$, and $k/h = 32 - 16h$. The limit of k/h as h approaches 0 is 32.
5. $\dfrac{h}{\sqrt{h+4} - 2} \cdot \dfrac{\sqrt{h+4} + 2}{\sqrt{h+4} + 2} = \dfrac{h\,(\sqrt{h+4} + 2)}{h}$. To obtain the limit as h approaches 0 we may divide numerator and denominator by h. This gives $\sqrt{h+4} + 2$. Now $\sqrt{h+4}$ approaches $\sqrt{4}$ so that the limit is $\sqrt{4} + 2$ or 4.
6. (a) No; they differ at $h = 0$.
   (b) Yes, because to determine the limit as h approaches 0 we consider the values of either function as h takes on values closer and closer to 0 but we do not consider the value $h = 0$.

CHAPTER 2, SECTION 5

1. Certainly as Δt becomes smaller and smaller, $16\Delta t$ becomes smaller and smaller. Moreover we can get $16\Delta t$ as close to 0 as we wish by taking Δt small enough. Specifically we have but to let Δt be $1/16$ of whatever small quantity we wish, to have $16\Delta t$ be that small quantity.
2. (a) $\Delta s = 16(3 + 1)^2 - 16 \cdot 3^2 = 112$.
   (b) $\Delta s = 16(4 + 1)^2 - 16 \cdot 4^2 = 144$.
   (c) $\Delta s = 16(4 + 1/2)^2 - 16 \cdot 4^2 = 68$.
3. $\Delta s = 16(5 + \Delta t)^2 - 16 \cdot 5^2 = 160\Delta t + 16(\Delta t)^2$. $\Delta s/\Delta t = 160 + 16\Delta t$ and the limit as Δt approaches 0 is 160.
4. $\Delta s = 16(t_1 + \Delta t)^2 - 16t_1^2 = 32t_1\Delta t + 16(\Delta t)^2$. $\Delta s/\Delta t = 32t_1 + 16\Delta t$ and the limit as Δt approaches 0 is $32t_1$.
5. $\Delta s = 2.6(4 + \Delta t)^2 - 2.6(4)^2 = 20.8\Delta t + 2.6(\Delta t)^2$. $\Delta s/\Delta t = 20.8 + 2.6\Delta t$ and the limit as Δt approaches 0 is 20.8.

6. $\Delta s = 432(5 + \Delta t)^2 - 432(5)^2 = 4320\Delta t + 432(\Delta t)^2$. $\Delta s/\Delta t = 4320 + 432\Delta t$ and the limit as Δt approaches 0 is 4320.
7. By taking Δt be 0.01 and calculating $\Delta s/\Delta t$.
8. Use the method of increments in each case and at the value of t stated. Thus
   (a) $\Delta s = 4(3 + \Delta t)^2 - 4(3)^2 = 24\Delta t + (\Delta t)^2$. $\Delta s/\Delta t = 24 + \Delta t$ and the limit as Δt approaches 0 is 24.
   (b) $3/2$. (c) 0. (d) 10.
9. A limit is a constant or fixed number.
10. A limit is an exact value.

CHAPTER 2, SECTION 6

1. In each case use formula (30). The answers not in the text are: b) 128, d) 160, f) 26, h) x_1.
2. Use formula (30). The answer not in the text is b) 60 ft/sec.
3. The third sentence merely rephrases the problem in mathematical terms. $A_1 = \pi r^2$, $A_1 + \Delta A = \pi(r_1 + \Delta r)^2$. Hence $\Delta A = 2\pi r_1 \Delta r + \pi(\Delta r)^2 = 2100\pi$ ft.
4. $y' = 2x_1$. Since for $y = ax^2$, $y' = 2ax$, the effect of the constant factor a in the function is that it multiplies the derivative of x^2.
5. $A = s^2$. Hence $A' = 2s_1$. The result is intuitively reasonable because it says that the area increases at the rate which is fixed by the lengths of two adjacent sides. As these sides increase the area of the whole square increases and the rate of increase depends on the lengths of these two sides. Compare the discussion of (28) in the text.
6. Since $A = \ell w$ and ℓ is kept fixed $dA/dw = \ell$. Geometrically, if ℓ is kept fixed and w changes by an amount Δw, the change $\Delta A = \ell\Delta w$. The instantaneous rate of change $\dot{A}$ depends then on the length of ℓ at the value w_1 of w at which the rate is computed.
7. Δy.
8. $\Delta y/\Delta x$.
9. $f'(x_0)$ or y' or dy/dx evaluated at x_0.
10. The derivative of $y = f(x)$ at $x = x_0$.
11. Δx is the independent variable and $\Delta y/\Delta x$ is the dependent variable.

CHAPTER 2, SECTION 7

1. $ds/dt = v = 10t$.
2. (a) The velocity (or instantaneous rate of change of distance with respect to time) at time t is $32t$.
   (b) Same as (a) except that $5t$ replaces $32t$.
   (c) The instantaneous rate of change of velocity with respect to time (or acceleration) at time t is -32.
   (d) The instantaneous rate of change of y with respect to x at any value of x is $4x$.

   Alternative characterizations are, for example: The derived function of s is $32t$, etc.
   (e) The instantaneous rate of change of y with respect to x is $8x$.
   (f) The instantaneous rate of change of $f(x)$ with respect to x is $-3x$.

3. (b) $\dot{s} = -6t$; (d) $dy/dx = 16x$; (f) $y' = -2x$; (h) $y' = 54x$; (j) $dy/dx = -15x$; (l) $y' = 2\sqrt{2}x$.

CHAPTER 2, SECTION 8

1. $y + \Delta y = b(x+\Delta x)$. Subtract $y = bx$. Then $\Delta y = b\Delta x$ and $\Delta y/\Delta x = b$. Hence since b is a constant, $\lim_{\Delta x \to 0} \Delta y/\Delta x = b$. That is, $dy/dx = y' = b$.
2. $y + \Delta y = c$. Subtract $y = c$. Then $\Delta y = 0$; $\Delta y/\Delta x = 0$. Hence $\lim_{\Delta x \to 0} \Delta y/\Delta x = 0$. That is, $dy/dx = y' = 0$.
3. b) $y' = -4x$, d) $\dot{d} = -9$, f) $\dot{s} = 0$.
4. $y_1 = ax_1^3$, $y_1 + \Delta y = a(x_1 + \Delta x)^3$, $\Delta y = 3ax_1^2\Delta x + 3ax_1(\Delta x)^2 + a(\Delta x)^3$, $\Delta y/\Delta x = 3ax_1^2 + (3ax_1 + a\Delta x)\Delta x$. Hence $y' = 3ax^2$.
5. $dV/dr = 4\pi r^2$, the surface area of a sphere of radius r.
6. The correct conjecture is that the derived function is obtained from the given function by multiplying by the exponent 4 and lowering the exponent by one to 3. Thus $y' = 4ax^3$. The method of increment gives $\Delta y/\Delta x = 4ax_1^3 + [6ax_1^2 + 4ax_1\Delta x + a(\Delta x)^2]\Delta x$ from which the result follows.
7. The formula for arc length is $s = r\theta$ where r is the radius of the circle and θ is the central angle in radians. Here $r = 5$; hence $s = 5\theta$ and $ds/d\theta = 5$.
8. Acceleration equals $\dot{v}$. Since $v = 32t$, $\dot{v} = 32$.
9. (a) $dy/dx = ny$, (b) $dy/dx = kx$, (c) $dy/dx = kx$, (d) $dA/dr = kr$.
10. The argument is false. The derivative of a function which is always constant (or of a constant term in a function) is 0. It is true that the derivative of any function is defined at a fixed value of x and the function is constant at the value of x but if the function varies as x changes the function is not a constant for all values of x and its derivative need not be 0.

CHAPTER 2, SECTION 9

1. (b) $\dot{s} = 20t$; (e) $y' = 2x-7$; (f) $y' = -2x+14$; (g) $y' = -7$; (h) $y' = 6\sqrt{2}x + \sqrt{3}$.
2. The method of increments gives $\Delta y/\Delta x = 2ax_1 + b$. Hence $dy/dx = 2ax + b$.
3. $f'(x) = 2x$; $f'(2) = 4$; $f'(-2) = -4$.
4. For both functions $y' = 2x$. We can argue that this must be the case for either of two reasons. We have some evidence to the effect that the derivative of $x^2 + 5$ is the sum of the derivatives of x^2 and 5 and the derivative of 5 is 0. The second reason is that the addition of a constant to x^2 does not change the rate at which x^2 varies because the constant does not change and so does not contribute to the rate of change.
5. $\dot{s} = 100 + 32t$. At $t = 4$, $\dot{s} = 328$ ft/sec.
6. At all values of x, T' or $dT/dx = -0.004$. Since T' is the limit of $\Delta T/\Delta x$ and T' is negative it must be that $\Delta T/\Delta x$ is negative. As x increases by the amount Δx, ΔT must then be negative. Hence the temperature must decrease as the altitude increases.
7. $\dot{x} = 18 - 6t$. Hence we get the answers to (a), (b) and (c) by letting t be 2, 3, and 4 respectively. As for (d), since the formula gives the amount being produced of the third substance a negative rate is physically meaningless. If after a certain value of t the combination of the two separate substances stops and thereafter the third substance decomposes into the original two and if the original formula holds for the decomposition a negative rate could mean physically the rate at which the third substance is decreasing because it is decomposing.
8. $dC/dx = 10x + 15$. At $x = 15$, $dC/dx = 165$. Yes.
9. $dC/dx = 6x - 4$. No.
10. $dP/dx = 4x - 6$. No. His profit may be declining when dP/dx is negative but he may still be making money.
11. $dC/dx = 6 - .004x$. When $x = 1200$, $dC/dx = 1.2$ dollars. A low marginal cost means a cheap cost of production and so is desirable.

CHAPTER 2, SECTION 10

1. (a) $\dot{s} = 32t + 100$; hence $\ddot{s} = 32$.
   (b) d^2s/dt^2 is an alternate notation for $\ddot{s}$, $d^2s/dt^2 = 32$.
   (c) $\ddot{s} = \dot{v} = 32$.
   (d) $\ddot{s} = 32$.
   (e) $\dot{v} = \ddot{s} = 32$.
2. $\ddot{s} = d^2s/dt^2 = \dot{v} = -32$. Same answer for all parts.
3. $\dot{v} = kv$.
4. $y'' = d^2y/dx^2 = 0$.
5. (b) $\ddot{s} = a = -32$; (d) $\ddot{y} = -6$; (e) $y'' = 8$; (g) $y'' = 0$; (h) $y'' - 2\sqrt{5}$;
   (j) $\ddot{s} = a = 32$.

Solutions to Chapter 3

CHAPTER 3, SECTION 1

1. (b) $y = 5x + C$; (d) $v = -32t + C$; (f) $y = \frac{3}{2}x^2 + C$.
2. (c) $s = -8t^2 + 50t + C$; (d) $y = \frac{3}{2}x^2 + 7x + C$; (f) $y = -(\sqrt{2}/2)\, x^2 + 8x + C$; (g) (g) $y = 50x - \frac{3}{2}x^2 + C$; (h) $y = 25x^2 - 3x + C$; (k) $s = 50t - (\sqrt{2}/2)\, t^2 + C$; (l) $s = (\sqrt{2}/2)\, t^2 + 50t + C$; (n) $s = -1.25t^2 + 1.6t + C$; (o) $y = \sqrt{3}t^2 + 5t + C$; (p) $y = \sqrt{6}x^2 - \frac{5}{2}\sqrt{3}x + C$.

CHAPTER 3, SECTION 2

1. Choose the downward direction as positive for distance traveled. Then $a = \dot{v} = 32$ and $v = 32t + C$. Measuring t from the instant of release, we have $v = 0$ when $t = 0$. Thus $C = 0$ and $v = \dot{s} = 32t$. Then $s = 16t^2 + C$. If s is measured from the point at which the object is released then $s = 0$ at $t = 0$. Thus $C = 0$ and $s = 16t^2$. When the object reaches ground, $s = 1000$; thus $t = \sqrt{1000/16} = 5\sqrt{10}/2$. The velocity at this time is $v = 32(\frac{5}{2}\sqrt{10}) = 80\sqrt{10}$ ft/sec.
2. As in No. 1, $v = 32t$ and $s = 16t^2 + C$. At the instant of release, the object is 50 feet in the positive direction (i.e., below the roof). Thus $s = 50$ when $t = 0$ and so $C = 50$. Then $s = 16t^2 + 50$. The distance fallen in t seconds is the position at time t minus the original position and so is equal to $(16t^2 + 50) - 50 = 16t^2$.
3. Distance is measured upwards. Hence $a = -32$, $v = -32t + C$. The balloon is stationary and the object is dropped. Hence $v = 0$ at $t = 0$. Then $C = 0$, $v = -32t$ and $s = -16t^2 + C$. Distance is measured from the ground upward; hence $s = 1000$ at $t = 0$. Then $C = 1000$, and $s = -16t^2 + 1000$.
4. As in exercise 1, $v = 32t + C$. Since $v = 0$ at $t = 10$ (the time of release), $C = -320$. Then $v = 32t - 320$. Hence $s = 16t^2 - 320t + C$. Since $s = 0$ at $t = 10$, $C = 1600$. Then $s = 16t^2 - 320t + 1600$.
5. As in Exercise 1, $v = 32t + C$. Since at $t = 0$, a positive velocity of 100 is imparted to the object, then when $t = 0$, $v = 100$. Then $C = 100$ and $v = 32t + 100$. Hence $s = 16t^2 + 100t + C$. Since $s = 0$ at $t = 0$, $C = 0$ and $s = 16t^2 + 100t$.
6. There is no acceleration. Then $a = 0$ and $v = \dot{s} = C$. Since the initial speed is 100, $v = 100$ when $t = 0$. Then $C = 100$ and $s = 100t + C$. If s is to be distance traveled then $s = 0$ at $t = 0$. Thus $s = 100t$.
7. As in Exercise 3, $a = -5.3$, $v = -5.3t$, $s = -(5.3/2)t^2 + C$. The object is at a height of 500 feet in the positive direction at $t = 0$. Thus $s = -(5.3/2)t^2 + 500$.
8. As in Exercise 3, $v = -32t + C$. At $t = 0$, a speed of 50 is imparted in the negative direction. Hence $v = -50$ at $t = 0$ and $C = -50$. Then $v = -32t - 50$ and $s = -16t^2 - 50t + C$. At $t = 0$, the height is 200 in the positive direction. Then at $t = 0$, $s = 200$ and so $C = 200$. Thus $s = -16t^2 - 50t + 200$.

9. This problem is a rewording of exercise 1.
10. Yes. By Exercise 1, the distance fallen in time T is $s = 16T^2$. The mean velocity, $\bar{v}$, between $t = 0$ and $t = T$ is $(0 + 32T)/2 = 16T$. With this mean or average velocity the distance traveled in T seconds is $16T \cdot T$ and this is the same result.
11. 45 miles per hour is 66 ft/sec. Now taking the positive direction of distance to be that in which the train is running, $a = -4/3$. Then $v = -4/3\,t + C$. If we measure time from the instant the brakes are applied then $v = 66$ when $t = 0$ and $C = 66$. Thus $v = -4/3\,t + 66$. Then $s = -2/3\,t^2 + 66t + C$. If we measure s from the point at which the brakes are applied then $s = 0$ when $t = 0$ and $C = 0$. Thus $s = -2/3\,t^2 + 66t$. The train runs until $v = 0$. This occurs when $t = 49.5$ sec. If we substitute this value of t in the expression for s we obtain $s = 1633.5$ ft.
12. For the first drop we have $v = 0$ and $s = 0$ at $t = 0$, thus $s = 16t^2$. For the second drop, $v = 32t + C$. But $v = 0$ when $t = 1$. Hence $C = -32$ and so $v = 32t - 32$. Now $s = 16t^2 - 32t + C$. At $t = 1$, $s = 0$. Hence $C = 16$. Then $s = 16t^2 - 32t + 16$. At $t = 2$ (i.e., 1 second after the second drop is released) we have $16 \cdot 2^2 - (16 \cdot 2^2 - 32 \cdot 2 + 16) = 48$ ft.
13. Denote the unknown acceleration by a. Measuring t and s from the time and point when the plane touches ground, $v = at + 100$ and $s = 1/2\,at^2 + 100t$. When the plane stops, $v = 0$, thus $t = -100/a$. At this time $s = 1/4$, hence we obtain the equation $1/4 = 1/2\,a(-100/a)^2 + 100(-100/a)$ for a. This yields $a = -20{,}000$, hence the deceleration is $20{,}000$ mi/hr². We could start with $-a$ and a positive and obtain the same result.
14. We measure time from the instant the first body is dropped. Since the body is dropped the time to fall 100 feet is given by $100 = 16t^2$ or $t = 5/2$. The second body is projected downward with some initial velocity v_0 at $t = 5/2$. We have for it $a = 32$; hence $v = 32t = C$. At $t = 5/2, v = v_0$. Hence $v_0 = 32(5/2) + C$ or $C = v_0 - 80$. Then $v = 32t + v_0 - 80$. Integrating gives $s = 16t^2 + v_0 t - 80t + C$. At $t = 5/2$, $s = 0$. Hence $C = -(5/2)v_0 + 100$ and $s = 16t^2 + v_0 t - 80t - (5/2)v_0 + 100$. Now when $t = 12.5$, the two distances fallen must be equal. Then

$$16(12.5)^2 = 16(12.5)^2 + v_0(12.5) - 80(12.5) - (5/2)v_0 + 100.$$

Hence $v_0 = 90$ ft/sec.

15. (a) Since $s = R\theta$, $\dot{s} = R\dot{\theta}$.

(b) From $\dot{s} = R\dot{\theta}$ we have $\ddot{s} = R\ddot{\theta} = R\alpha$.

(c) $\ddot{\theta} = \alpha$. Then $\dot{\theta} = \alpha t + C$. Since $\dot{\theta} = 0$ when $t = 0$ then $C = 0$. Then $\dot{\theta} = \alpha t$ and $\theta = (\alpha/2)t^2 + C$. Let θ be measured from the position of the object at $t = 0$. Then $\theta = 0$ when $t = 0$ and $\theta = (\alpha/2)t^2$. When $t = 120$ sec., $\theta = 3600 \cdot 2\pi$ radians. Then $7200\pi = (\alpha/2)(120)^2$ or $\alpha = \pi$ rad/sec^2.

(d) As in part (c), $\dot{\theta} = \alpha t$ and $\theta = (\alpha/2)t^2$. Here when $t = 5$, $\theta = 12.5(2\pi)$. Then $\alpha = 2\pi$ rad/sec^2 and at $t = 5$, $\dot{\theta} = 2\pi \cdot 5 = 10\pi$ rad/sec.

(e) $\dot{\omega} = \alpha$, $\omega = \alpha t + C$. At $t = 0$, $\omega = 2\pi$, hence $\omega = \alpha t + 2\pi$. The wheel comes to rest when $\omega = 0$ and $t = -2\pi/\alpha$. Now $\omega = \dot{\theta}$. Hence $\theta = \frac{1}{2}\alpha t^2 + 2\pi t + C$. If we measure θ from the instant when the friction applies $(t = 0)$ then $C = 0$ and $\theta = (\alpha/2)t^2 + 2\pi t$. The wheel stops when $t = -2\pi/\alpha$ and $\theta = 10 \cdot 2\pi$. Hence an equation for α is $10 \cdot 2\pi = \frac{1}{2}\alpha(-2\pi/\alpha)^2 + 2\pi(-2\pi/\alpha)$ or $\alpha = -\pi/10$.

(f) $\dot{\omega} = \alpha$. Hence $\dot{\theta} = \alpha t$, $v = R\dot{\theta} = 1 \cdot \alpha t$. For $t = 10$ min $= 600$ sec, $v = 100$ ft/sec. Hence $100 = 600\alpha$ or $\alpha = \frac{1}{6}$ ft/sec^2. Then $v = \frac{1}{6}t$ and $\dot{\theta} = \frac{1}{6}t$. At $t = 15$, $v = \frac{5}{2}$ ft/sec and $\dot{\theta} = \frac{5}{2}$ rad/sec.

CHAPTER 3, SECTION 3

1. Formula (27) gives the height above ground. Thus at $t = 5$ this height is 240 ft. However, at $t = 5$, the object is headed downward; hence the distance traveled is the maximum height plus the distance traveled downward after reaching the maximum height or 272 ft (compare figure 3-3).
2. Follow the derivation of (27) except that 160 replaces 128.
3. Let $s = 512$ and solve for t. We obtain $t = 4$ and 8. The object is at a height of 512 ft both on its ascent and on its descent.
4. $v = \dot{s} = 144 - 32t$; hence at $t = 9$, $v = -144$ ft/sec. But at $t = 9$, $s = 0$. Hence the object has just returned to the ground.
5. As in the derivation of (25) and (27) $v = -32t + 1000$, $s = -16t + 1000t$. Maximum height occurs when $v = 0$, i.e., at $t = {}^{1000}/_{32}$ and at this value of t, $s = 15{,}625$ ft.
6. Here $a = \dot{v} = -5.3$ if we take the upward direction as positive. Then $v = -5.3t + C$ and since $v = 96$ when $t = 0$, $v = -5.3t + 96$. Then $s = -(5.3/2)t^2 + 96t + C$. If s is measured from the surface then $s = 0$ when $t = 0$. Hence $C = 0$ and $s = -(5.3/2)t^2 + 96t$. At the maximum height $v = 0$ or $t = 96/5.3 = 18.1$ sec. At this value of t, $s = 869.5$ ft.
7. By following the derivation of (27) except that 96 replaces 128 we have $v = -32t + 96$ and $s = -16t^2 + 96t$. When the stone reaches the ground $s = 0$. Hence $t = 0$ and $t = 6$. The value $t = 6$ is the value at which the stone hits the ground. At $t = 6$, $v = -96$. Note that this is the same as the initial velocity but of opposite sign.
8. Follow the derivation of (27) except that v_0 replaces 128 and g replaces 32.
9. Use the result of Ex. 8 except that $g = 32$. Thus $s = v_0t - 16t^2$. When $t = 20$, $s = 0$. Hence $v_0 = 320$ ft/sec.
10. $\dot{v} = -32$ and so $v = -32t + C$. At $t = 5$, $v = 200$. Hence $C = 360$. Then $v = -32t + 360$ and $s = -16t^2 + 360t + C$. If s is measured from the ground up, $s = 0$ when $t = 5$. Then $C = -1400$. Hence $s = -16t^2 + 360t - 1400 = -16(t - 5)^2 + 200(t - 5)$.
11. For the first object if we follow the derivation of (27) except that 200 replaces 128 we have $s = -16t^2 + 200t$. For the second object, $\dot{v} = -32$ and and so $v = -32t + C$. Here at $t = 5$, $v = 300$. Hence $C = 460$. Then $v = -32t + 460$. Now $s = -16t^2 + 460t + C$. At $t = 5$, $s = 0$. Then $C = -1900$ and $s = -16t^2 + 460t - 1900$. When the two objects meet, the two s-values are equal. Then $t = 7.3$ sec approx. This is the number of seconds after the first object is thrown up. If we substitute this value of t in either formula for s we obtain 606 ft approx.

12. As in the derivation of (25) and (27) except that v_0 replaces 128 we have $v = -32t + v_0$ and $s = -16t^2 + v_0 t$. The height of 1000 ft is to be the maximum height. There $v = 0$ so that $t = v_0/32$. At this value of t, s is to be 1000. Hence $1000 = -16(v_0/32)^2 + v_0(v_0/32)$. Hence $v_0 = 80\sqrt{10}$ ft/sec.
13. This Exercise is just a rewording of Exercise 12.
14. As in the derivation of (25) and (27) except that V replaces 128 we obtain $v = -32t + V$, $s = 16t^2 + Vt$. At the maximum height $v = 0$ and $t = V/32$. Then $s = -16(V/32)^2 + V(V/32) = V^2/64$.
15. We start with $\dot{v} = 8$ so that $v = 8t + C$. Since the train starts from rest $v = 0$ when $t = 0$ so that $C = 0$. Then $v = 8t$ and $s = 4t^2 + C$. If we measure s from the first station $s = 0$ when $t = 0$ and so $s = 4t^2$. When $v = 20$, $t = \frac{5}{2}$ and $s = 25$. During the second portion of the trip which lasts for some unknown time t_1, the additional distance covered is $s_1 = 20t_1$. The deceleration stage of the trip is best treated separately. We have $\dot{v} = -12$ and $v = -12t + C_1$. Let us measure t from the instant the deceleration begins. Then when $t = 0$, $v = 20$ and $C = 20$. Thus $v = -12t + 20$ and $s = -6t^2 + 20t + C$. If we measure this s from the point where the deceleration begins then $s = 0$ when $t = 0$ and so $C = 0$. Then $s = -6t^2 + 20t$. The train comes to rest when $v = 0$ and so $t = \frac{5}{3}$ and $s = -6(\frac{5}{3})^2 + 20(\frac{5}{3}) = \frac{50}{3}$. Thus the total distance traveled is $25 + 20t_1 + \frac{50}{3}$ and this equals 400 ft. Then $t_1 = \frac{215}{12}$. The total time traveled is $\frac{5}{2} + \frac{215}{12} + \frac{5}{3} = 22\frac{1}{12}$ sec.
16. After the fuel is exhausted, $a = -32$, $v = -32t + C$. If time is measured from the instant the fuel is exhausted, $v = 176$ when $t = 0$. Then $C = 176$ and $v = -32t + 176$. Then $s = -16t^2 + 176t + C$. If the distance is measured from the ground then $s = 52800$ when $t = 0$. Thus $C = 52800$ and $s = -16t^2 + 176t + 52800$.
17. (a) We follow the derivation of (25) and (27) except that 96 replaces 128 and height is measured from the roof. Then $s = -16t^2 + 96t$.
    (b) When the ball reaches the ground $s = -112$. For this value of s, $t = 7$.
18. (a) $v = -32t + 96$. Then $s = -16t^2 + 96t + C$. If height is measured from the ground and since t is already measured from the instant the ball is thrown up, $s = 112$ when $t = 0$. Then $C = 112$.
    (b) When the ball reaches the ground $s = 0$. Since $s = -16t^2 + 96t + 112$, when $s = 0$, $t = 7$. The other root $t = -1$ has no physical significance here.
19. $a = \dot{v} = -11$. Then $v = -11t + C$. If t is measured from the instant deceleration begins then $v = 88$ when $t = 0$ and so $C = 88$. Then $v = -11t + 88$ and $s = -\frac{11}{2}t^2 + 88t + C$. If distance is measured from the point where deceleration begins then $s = 0$ when $t = 0$ and $C = 0$. Then $s = -\frac{11}{2}t^2 + 88t$. When the object comes to rest, $v = 0$. Then $t = 8$ and for this t, $s = 352$ ft.
20. We found in the text that it takes 4 seconds for the ball to reach maximum height. When it reaches the ground $s = 0$. From (29) we have $0 = -16t^2 + 128t$ or $t = 8$. Hence it takes 4 seconds to go from maximum height to the ground.

21. After braking $\dot{v} = -a$ and $v = -at + C$. If time is measured from the instant the brakes are applied, $v = v_0$ when $t = 0$. Then $v = -at + v_0$ and $s = -at^2/2 + v_0t + C$. If distance is measured from the point where the brakes are applied, $s = 0$ when $t = 0$ and so $s = -at^2/2 + v_0t$. The car stops when $v = 0$ or $t = v_0/a$. Then for this t, $s = -(a/2)(v_0/a)^2 + v_0(v_0/a) = v_0^2/2a$. During the one second of reaction time the car travels $v_0\tau$.

22. Here $a = -5$ and $v = -5t + C$. If time is measured from the instant of braking $v = 44$ when $t = 0$ and so $v = -5t + 44$. Then $s = -5t^2/2 + 44t + C$. If distance is measured from the point at which the brakes are applied then $s = 0$ when $t = 0$ and $s = -5t^2/2 + 44t$. When the car stops $v = 0$ and $t = {}^{44}/_5$. In this time the distance traveled is 193.6 ft.

23. During the first part of the $a = f$, $v = ft$ and $s = ft^2/2$ if time is measured from the beginning of the trip and distance likewise. For the second part of the trip, $a = -r$ and $v = -rt + C$. Suppose the first part of the trip lasts t_1 seconds. Then when $t = t_1$, $v = ft_1$ and so $C = (f + r)t_1$. Then $v = -rt + (f + r)t_1$ and $s = -rt^2/2 + (f + r)t_1t + C$. When $t = t_1$, $s = ft_1^2/2$. Then $C = -(t_1^2/2)(f + r)$. When the trip ends $v = 0$ or $-rt + (f + r)t_1 = 0$. Now t is specified so that $t_1 = rt/(t + r)$. If we substitute this value of t_1 in the expression for s we obtain $s = [fr/(f + r)]t^2/2$.

24. (a) $a = -32$ and so $v = -32t + C$. When $t = 0$ (the instant the bomb is released) $v = 1500$. Hence $C = 1500$ and $v = -32t + 1500$. Then $s = -16t^2 + 1500t + C$. If we measure height from the ground then $s = 10000$ when $t = 0$ and so $s = -16t^2 + 1500t + 10000$. When the bomb reaches the ground $s = 0$. If we solve for t we obtain about 100 sec.
(b) After 100 sec the bomber will be 120,000 ft. away horizontally from the point at which the bomb is released and 10,000 ft above the ground. Hence its distance from the bomb will be given by the Pythagorean theorem, that is, $\sqrt{(120{,}000)^2 + (10{,}000)^2}$ or about 24 miles.

25. $x = 9t - (t^3/6) + C$ and since $x = 0$ when $t = 0$, then $C = 0$. At $t = 4$, $x = 25\frac{1}{3}$ gms.

26. For the first body, if time is measured from the instant it is dropped and distance from the ground up, $a = -32$, $v = -32t$, $s = -16t^2 + 300$. For the second body, $a = -32$, $v = -32t + C$ and since $v = 120$ when $t = 2$, $C = 184$ and $v = -32t + 184$. Then $s = -16t^2 + 184t + C$. For this second body $s = 0$ when $t = 2$ and so $C = -304$. Then $s = -16t^2 + 184t - 304$. When the two bodies meet their s-values are equal. Hence $-16t^2 + 300 = -16t^2 + 184t - 304$. Then $t = 3.3$ sec approx. (after the first body is dropped). For this t either formula for s gives 126 ft. approx.

27. Take the downward direction as positive. $a = 32$ so that $v = 32t + C$. When $t = 0$, $v = -20$ and so $C = -20$. Then $v = 32t - 20$ and $s = 16t^2 - 20t + C$. If s is measured from the balloon then at $t = 0$, $s = 0$ and $C = 0$. Then $s = 16t^2 - 20t$. When $t = 6$, $s = 456$ ft.

28. We can use the general result of Exercise 8, namely, $s = v_0t - gt^2/2$. If we select any value s_0 of s and solve for t we obtain $t = (v_0 \pm \sqrt{v_0^2 - 2gs_0})/g$. But $v = v_0 - gt$. If we substitute the two values of t in the formula for v we obtain $v = \pm\sqrt{v_0^2 - 2gs_0}$. The numerical values of v are the same. Note that

a relative maximum; at $x = -1$ there is a relative maximum; at $x = 1$ there is a relative maximum.

(g) $f(x) = x^4$; $f'(x) = 4x^3$. Hence $x = 0$ is a possible value. $f'(x)$ does change from negative to positive around $x = 0$. Hence at $x = 0$ there is a relative minimum.

(h) $f(x) = x + 1/x$; $f'(x) = 1 - 1/x^2$. Hence $x = 1$ and $x = -1$ are possible values. For x slightly less than -1, say $-5/4$, $f'(x)$ is positive and for x slightly more than -1, say $-3/4$, $f'(x)$ is negative. Hence there is a relative maximum at $x = -1$. At $x = 1$, $f'(x)$ changes from negative to positive; hence there is a relative minimum at $x = 1$.

(i) $f(x) = x\sqrt{x-1}$; $f'(x) = (x/2\sqrt{x-1}) + \sqrt{x-1}$. Now $f'(x) = 0$ at $x = 2/3$. But the function has no real value at $x = 2/3$. Hence there are no maxima and minima.

(j) $f(x) = x^2/(x-1)$; $f'(x) = (x^2-2x)/(x-1)^2$. Possible values are $x = 0$ and $x = 2$. At $x = 0$, $f'(x)$ changes from positive to negative. Hence there is a relative maximum there. At $x = 2$, $f'(x)$ changes from negative to positive. Hence there is a relative minimum there.

2. (a) $y' = -2x + 6$. Hence $x = 3$ is a possible relative maximum or minimum. At $x = 3$, y' changes from + to −. Hence there is a relative maximum whose value is 16. The absolute maxima and minima may occur at the end values 0 and 5. At $x = 0$, $y = 7$ and at $x = 5$, $y = 12$. Hence the relative maximum of 16 is also the absolute maximum and the absolute minimum is 7.

(b) $y' = 3x(x-2)$. At $x = 0$, y' changes from + to −. Hence there is a relative maximum of 4. At $x = 2$, y' changes from − to +. Hence there is a relative minimum of 0. At $x = -2$, $y = -16$. This is the absolute minimum. At $x = 4$, $y = 20$. This is the absolute maximum.

3. $y' = 2(x-1)(x+1)^2 + 2(x+1)(x-1)^2 = 4x(x+1)(x-1)$. At $x = -1$, y' changes from − to +. Hence there is a relative minimum and its value is 0. At $x = 0$, y' changes from + to −. Hence there is a relative maximum whose value is 1. At $x = 1$, the behavior of y' is as at $x = -1$ and the relative minimum is again 0.

4. Here $y' = -2/3(x-1)^{1/3}$. This y' is never 0. However the function may have a relative maximum or minimum where the derivative fails to exist. We see that $(x-1)^{2/3}$ is positive for every value of x and this is subtracted from 3. The least we can subtract is 0 and this occurs when $x = 1$. Then $y = 3$ is a relative maximum. As x increases or decreases from the value of 1, y continually decreases.

5. (a) The function $y = x$ is an example. However $y = -x^2$ in the interval from $-\infty$ to 0 is a better example. Here as x increases, $f'(x)$ actually de-

s_0 must be a value actually attained by the object or the values of t and of v will be complex.

29. Again from Exercise 8, $v = v_0 - gt$ and at the maximum height $v = 0$ so that $t = v_0/g$. Since $s = v_0t - gt^2/2$ we find that the maximum height is $v_0^2/2g$. To attain twice this maximum height we must replace v_0 by $\sqrt{2}v_0$.

CHAPTER 3, SECTION 4

1. We use (35) to calculate the time. In (35) $s = 200$ and $A = 30°$. Hence $200 = 8t^2$ and $t = 5$ sec. To calculate the velocity we use (34) wherein $A = 30°$ and $t = 5$. Hence $v = 80$ ft/sec.
2. If s is the distance the object slides, then $\sin 30° = 100/s$ or $s = 200$ ft. The rest is the same as in Exercise 1.
3. The only change over Exercise 2 is that A is $15°$. Hence $\sin 15° = 100/s$. Since $\sin 15° = .2588$, $s = 386$. Now use (35) with $s = 386$ and $A = .2588$ to calculate t. The result is approximately 10 sec. We then use (34) with $\sin A = .2588$ and $t = 10$. Here if the value of t were calculated very accurately we would get the same 80 ft/sec as in Ex. 1. Note that the velocity at the bottom is the same if the height from which the object descends is the same. See Ex. 6(b) below.
4. Yes.
5. (a) Here $\sin A = h/\ell$. We use (35) in which $\sin A = h/\ell$ and $s = \ell$. Then $\ell = 16t^2h/\ell$ or $t = \ell/4\sqrt{h}$.
   (b) We use (34) in which $\sin A$ is now h/ℓ and $t = \ell/4\sqrt{h}$. Then $v = 8\sqrt{h}$.
6. (a) By Ex. 5 (a), $t_1 = \ell_1/4\sqrt{h}$ and $t_2 = \ell_2/4\sqrt{h}$. Then $t_1/t_2 = \ell_2/\ell_1$.
   (b) By Ex. 5 (b) both velocities are $8\sqrt{h}$.
7. From the formula $s = 16t^2$ we find that the time to fall the distance OP is $t_1 = \sqrt{OP}/4$. For the motion along OP' we use (35), that is, $s = 16t^2 \sin A$. The distance OQ that the object slides in time t_1 is $OQ = 16(OP/16)\sin A$. Then $\sin A = OQ/OP$. Suppose Q is not on the circle but R on OP' is. Then $\measuredangle OPR$ is $\measuredangle A$ by the use of right triangles. Then $\sin A = OR/OP$. But $\sin A = OQ/OP$. Hence $Q = R$ and Q lies on the circle.
8. At any given time t the bead which falls straight down falls some distance OP. By Ex. 7, Q lies on a circle with OP as diameter. But the argument in Ex. 7 was not restricted to any specific angle A. Hence Q', Q'', ... all lie on the same circle.
9. If we consider circles with O as highest point (so that the diameters are all vertical line segments from O downward) the smallest of these circles which reaches C is the one which first touches it as the circles expand from O. The time for a bead to fall straight down will be least for this circle as compared with larger ones because for any one circle there is a time value to reach it and the time increases with the diameter (See Ex. 7). In this time the object sliding from O to Q will reach Q.
10. We start with (33), namely, $a = 32 \sin A$. Then $v = 32t \sin A + C$. At $t = 0$, $v = v_0$. Then $C = v_0$ and $v = 32t \sin A + v_0$. Integrating gives $s = 16t^2 \sin A + v_0t + C$. At $t = 0$, $s = 0$. Hence $s = 16t^2 \sin A + v_0t$.

CHAPTER 3, APPENDIX, SECTION 2

1. (b) $\sqrt{89}$; (d) $\sqrt{40}$.
2. Formula (1) is unaltered if x_2 and x_1 and y_2 and y_1 are interchanged.
3. AB = 10; BC = $\sqrt{125}$; AC = $\sqrt{125}$.
4. The lengths are $\sqrt{34}$; $\sqrt{34}$ and $\sqrt{36}$.
5. If we use Fig. 3A-1, locate the midpoint (x_3, y_3) on the line segment P_1P_2, and draw y_3 we find that y_1, y_3 and y_2 are parallel lines and y_3 cuts P_1P_2 in half. Then it must cut the other transversal RS in half and so $x_3 = [(x_2 - x_1)/2] + x_1 = (x_1 + x_2)/2$. Further y_3 is the median of the isosceles trapezoid RP_1P_2S and so is $(y_1 + y_2)/2$.

CHAPTER 3, APPENDIX, SECTION 3

1. Formula (2) applies in each case:
(b) $5/6$; (d) $-5/8$; (f) No slope.

CHAPTER 3, APPENDIX, SECTION 4

1. In each case we have but to find the tangent of the given angle.
2. In each case we have but to find the angle whose tangent is given.
3. The slope of each line is found by using formula (2) and we then find the angle A whose tangent is the slope.

CHAPTER 3, APPENDIX, SECTION 5

1. By using (2) we find that both lines have a slope of $1/6$ and so are parallel.
2. The slope of the first line is $3/2$ and the slope of the second one is $-2/3$. Each slope is the negative reciprocal of the other.
3. The given line has slope 1; then the perpendicular line has slope -1 or inclination $135°$.
4. The slope of the perpendicular is $-1/2$ and the inclination is the angle whose tangent is $-1/2$ or the angle whose supplement is $26°30'$ (approx.). Hence $153°30'$.
5. The given line has slope $-6/7$. Hence any perpendicular has slope $7/6$ or 1.1666. Then the inclination is $59°$ (approx.).

6. If two sides have slopes which are negative reciprocals of each other, the triangle is a right triangle. The slopes are (a) $9/4$, $4/9$, -1; (b) $3/4$, $-4/3$, 7; (c) $1/2$, $-6/5$, $-4/9$; (d) $(-5\sqrt{3}-5)/(5\sqrt{3}-5)$, $(-5+5\sqrt{3})/(-5-5\sqrt{3})$, 1. Thus only (b) is a right triangle.

9. The slopes of the sides are $3/7$, -2, $3/7$, -2.

CHAPTER 3, APPENDIX, SECTION 6

1. (a) Use formula (7) with $m_2 = 4$ and $m_1 = 3$. Then find θ.
   (b) Since $\tan 30° = \sqrt{3}/3$ and $\tan 135° = -1$, let $m_1 = \sqrt{3}/3$ and $m_2 = -1$, and use (7). Then $\tan\theta = -3.7$ approx. and $\theta = 105°$. The result can be obtained from a figure at once.
   (c) Let $m_2 = -2$ and $m_1 = 3$ and use (7).
2. The first line has slope 1 and the second, $-4/3$. Let $m_2 = -4/3$ and $m_1 = 1$ and use (7).

CHAPTER 3, APPENDIX, SECTION 7, FIRST SET

1. (a) Use (8).
   (b) We substitute 2 for x and 3 for y in $y - 2 = 5(x - 1)$. The equation is not satisfied and so $(2, 3)$ does not lie on the line.
2. For (a), (b), and (c) use (8) and merely substitute the given values.
   (d) use (8) or (9). In (9) b is 5.
3. We find the equation of the line determined by $(3, 6)$ and $(4, 7)$. The slope is 1 and so the equation is $y - 6 = 1(x - 3)$. To show that $(5, 8)$ lies on this line we substitute 5 for x and 8 for y. Then $8 - 6 = 1(5 - 3)$ and the equation is satisfied.
4. (a) The desired line must also have slope 2. Now use (8).
   (b) The desired line must have slope $-1/2$. Hence by (8), $y - 2 = -1/2(x - 4)$.
5. Since the given lines, by (9), have slopes 3 and -2, now use (7) with $m_1 = 3$ and $m_2 = -2$.

CHAPTER 3, APPENDIX, SECTION 7, SECOND SET

1. (a) We solve the given equation for y; then $y = -2/3x - 5/3$. By comparison with $y = mx + b$ we have the answers.
   (b) As in (a), $y = 3/4x - 7/4$. Then $m = 3/4$ and $b = -7/4$.
   (c) As in (a), $y = -3/4x$. Then $m = -3/4$ and $b = 0$.
2. Parallel to the x-axis and 3 units above it.
3. The line $3x + y + 7 = 0$ has slope -3. Hence the desired line is $y - 2 = -3(x - 1)$.
4. The given line has slope -3. Hence the desired line has slope $1/3$. Then the desired equation is $y + 1 = 1/3(x - 5)$.
5. Since $(0, 0)$ must lie on $Ax + By + C = 0$, $A \cdot O + B \cdot O + C = 0$.
6. The slope of the first line is $-A/B$ and of the second $-a/b$. Then the equality of these two slopes gives the result.
7. The slopes of the two given lines are $-2/3$ and $3/2$. These slopes are negative reciprocals.
8. The slope is $-2/3$. Hence find θ for which $\tan\theta = -2/3$. Hence $\theta = 146°\ 20'$ (approx.).
9. The two given lines have slopes of $-3/2$ and $1/7$. Use (7) with $m_2 = -3/2$ and $m_1 = 1/7$.

10. A method is given in the problem. Only in (a) do the three given points lie on one line.
11. (a) $-A/B = 3/2$; (b) $A = (-1/4)C$ and $B = -1/3C$; (c) $C = 0$; (d) $A = 0$; (e) $B = 0$.
12. The equation of a curve (including straight lines) is an equation involving x, y and constants. For a fixed curve only x and y can vary. In the equation $x^2 + y^2 = r^2$, r must change with x and y if any given x and y are substituted in the equation. Hence r is not fixed as we choose different (x, y)'s along the straight line.

CHAPTER 3, APPENDIX, SECTION 8

1. In each case we use formula (14). Thus for (b), $x_1 = 3$, $y_1 = 2$, $A = 4$, $B = -1$, and $C = 2$.

Solutions to Chapter 4

CHAPTER 4, SECTION 3

2. In each case find the derivative at the given value of the independent variable. Thus
   (a) $\dot{s} = 8t$ and at $t = 2$, $\dot{s} = 16$.
   (b) 4; (c) 2; (d) 0.
3. In each case find the slope of the tangent and since $m = \tan A$, then find A. Thus
   (a) $y' = 2x$. At $x = 3$, $y' = 6$. Then $\tan A = 6$ and $A = 80° 32'$ (approx.).
   (b) $89° 7'$; (c) $104° 2'$; (d) $0°$; (e) $0°$.
4. The direction of motion is given by the slope or the inclination of the tangent. Since $y' = -8x + 16$, at $x = 3$, $y' = -8$. Then the inclination is $97° 7'$. These values also distinguish the sense of the motion; that is, the notion of slope presupposes that we consider whether the line rises or falls as we go from left to right. Hence the object is moving downward at $x = 3$.
5. $y' = -8 + 16$. At $x = 2$, $y' = 0$. Thus at this point the object is moving horizontally, i.e., perpendicular to the wall and so direct impact.
6. (a) $y' = 2x$. Thus the tangent thru $x = -3$, $y = 9$ has slope -6. Hence the equation of the tangent is $y - 9 = -6(x + 3)$ or $y = -6x - 9$.
   (b) $y = -2x - 4$.
7. The slope is given by the derivative. For (a) $y' = x/50$. At $x = 3$, $y' = 3/50$.
   (b) $x = 5$.
8. The value of the slope of the tangent to the curve at any point x is given by $2x$. This function does not have the same value for two distinct values of x.
9. Since $y' = 3x^2$, the values of y' at $x = 0$, 1, and -1 are 0, 3, 3.
10. The slope of the tangent at any point x is $3x^2$. At $x = 4$, the slope is 48; this is also the slope at $x = -4$ or at the point $(-4, -64)$.
11. The derivative is positive at $x = a$, decreases to zero at the crest, then becomes negative, then increases to zero at the trough, and then becomes positive.
12. At $t = 0$; at $t = b$; at maximum point.
13. The two curves are the same except that one is 5 units above the other. At any given x value their tangents are parallel.
14. (a) $y' = x$; (b) $y = k(y')^2$ (c) $y' = \frac{1}{2}(y/x)$.

CHAPTER 4, SECTION 4

1. (a) $p = 5/2$; hence $y = (1/10)x^2$; (b) $p = 1.5$; hence $y = (1/6)x^2$

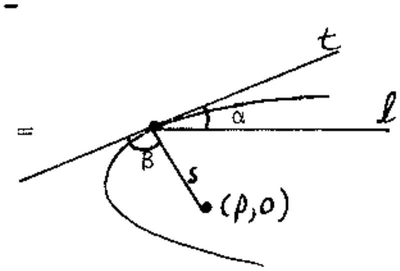
2. Compare formula (9). Then $p = 3$. Hence the focus is $(0,3)$ and the directrix is $y = -3$.
3. Since the focus is $(0,3)$ and the right-hand point on the parabola at that height is $(x,3)$, we let $y = 3$ in $y = \frac{1}{12}x^2$ and obtain $x = 6$. The width across the entire parabola at the height of the focus is then 12. For any parabola of the form $y = (1/4p)x^2$, the focus is at $(0,p)$. The points on the parabola at the same height are obtained by letting $y = p$ in the equation so that $x = \pm 2p$. Then the width of the parabola is $4p$.
4. One would expect that the x and y axes are interchanged in (9).
5. One would expect that y should be replaced by $-y$ in (9).
6. (a) $y = (1/16)x^2$, $p = 4$, focus $(0,4)$, directrix $y = -4$, latus rectum 16.
   (b) $x = (-1/8)y^2$, $p = 2$, focus $(-2,0)$, directrix $x = 2$, latus rectum 8.
   (c) $x = (2/9)y^2$, $p = 9/8$, focus $(9/8,0)$, directrix $x = -9/8$, latus rectum $9/2$.
   (d) $y = (-3/4)x^2$, $p = 1/3$, focus $(0,1/3)$, directrix $y = -1/3$, latus rectum $4/3$.
7. (a) Form is $x = (1/4p)y^2$, $p = 5$.
   (b) Form is $y = -(1/4p)x^2$, $p = 2$.
   (c) Form is $x = (1/4p)y^2$, $p = 1/2$.
8. To obtain the equation directly let (x,y) be any point on the parabola. Then $\sqrt{(x-0)^2 + (y-0)^2} = y + 2p$. Then $y = (1/4p)x^2 - p$. If we compare Fig. 4-15 with Fig. 4-12 we see that the y-values in Fig. 4-15 are p less than those in Fig. 4-12.
9. If (x,y) are the coordinates of any point on the parabola then $\sqrt{(x-4)^2 + (y-0)^2} = y + 8$. Simplifying gives the answer in the text.
10. (a) All the y-values of $y = x^2 + 6$ are 6 units above those of $y = x^2$.
    (b) Each y-value of $y = 3x^2$ is 3 times as large as the y-value of $y = x^2$ for the same x-value.
    (c) If we write $y = (x+6)^2$ as $y = x'^2$ where $x' = x + 6$ we see that $x = x' - 6$. Then each x-value is 6 units to the left of the x'-value. Thus $y = (x+6)^2$ is obtained from $y = x'^2$ by shifting or translating the latter 6 units to the left.
11. (a) The tangent line at $x = x_0$ is $y - (x_0^2/4p) = (x_0/2p)(x - x_0)$. This line cuts the x-axis at $x = x_0/2$ because $y = 0$ there, and we solve for x.
    (b) Draw the straight line between the points $(x_0, x_0^2/4p)$ and $(x_0/2, 0)$.
12. Choose axes so that the origin is at the lowest point or vertex of the parabola. Then one point on the parabola is $x = 1400$ and $y = 148$. Since the form of the equation is $y = (1/4p)x^2$, substitute 1400 for x and 148 for y to determine p. Then $y = 37x^2/490{,}000$.
13. If the axes are chosen so that the origin is at the center (top) of the arch, the equation of the arch is of the form $y = -x^2/4p$. Since the point $(25,-25)$ lies on the arch, $4p = 25$ and the equation of the arch is $y = -x^2/25$. The right-hand wheels of the truck are at $x = 23$. Then the y-value of the point on the arch whose x value is 23 is $-21\frac{4}{25}$. The roadway lies along $y = -25$. Hence the clearance is only $3\frac{21}{25}$ feet, whereas the truck needs 10 ft.

14. Compare Fig. 4-17 with Fig. 4-12. Then $p = 15$; hence $y = x^2/60$.
15. Choose the coordinate system so that the parabola is given by $y = (1/4p)x^2$. The tangent at the vertex is $y = 0$ or the x-axis. The tangent at any point (x_0, y_0) on the parabola has slope $x_0/2p$. The perpendicular from the focus to any tangent line has slope $-2p/x_0$. The equation of the tangent line is $y - y_0 = (x_0/2p)(x - x_0)$ and the equation of the perpendicular is $y - p = (-2p/x_0)(x - 0)$. We solve these last two equations simultaneously for their point of intersection. To do this take the value of x from the second equation and substitute it in the first one. If we use the fact that $y_0 = x_0^2/4p$ we find that $y = 0$ or the two lines intersect on the x-axis.
16. Assume that the coordinate system is as in Exercise 12. The tangent line at $x = x_0$ is $y - (x_0^2/4p) = (x_0/2p)(x - x_0)$. Similarly the tangent at $x = x_1$ is $y - x_1^2/4p = (x_1/2p)(x - x_1)$. If these lines are perpendicular then $x_1/2p = -2p/x_0$ or $x_1 = -4p^2/x_0$. Now solve the first two equations simultaneously and use the relation between x_0 and x_1. When solving take the value of x from one equation and substitute in the other. All we need to show is that the y-value of the point of intersection is $-p$.
17. (a) Let P have coordinates $(x_0, x_0^2/4p)$. The tangent P is $y - x_0^2/4p = (x_0/2p)(x - x_0)$. The point Q has coordinates $(0, x_0^2/4p)$ while T (the intersection of the tangent with the y-axis has coordinates $(0, -x_0^2/4p)$. Thus the midpoint of $\overline{QT}$ has coordinates $(0,0)$ which is the vertex.
    (b) The normal has slope $-2p/x_0$ and passes thru $(x_0, x_0^2/4p)$. Thus the equation of the normal is $y - x_0^2/4p = (-2p/x_0)(x - x_0)$. It then follows that the coordinates of R are $(0, x_0^2/4p + 2p)$ and hence $\overline{RQ}$ has length 2p.

CHAPTER 4, SECTION 5, FIRST SET

1. In finding the slope of the tangent line to the parabola.
2. No. We showed merely that the parabola does have the reflection property.
3. Suppose (Fig. 4-23) FP is a ray starting from the focus. Then PD is the reflected ray and this is parallel to the x-axis. Likewise QD′ is parallel to the axis. Thus $\angle D'QF + \angle DPF = 180°$. Then the sum of the supplement of $\angle D'QF$ and the supplement of $\angle DPF$ is 180°. By the law of reflection one-half of this sum lies inside the triangle QPR, where R is the intersection of the tangents. Then the remaining angle at R must also be 90°.

4. It is better to take the parabola in the position shown here. Then m_1 = slope of $\ell = 0$; m_2 = slope of $t = 2p/y_1$; m_3 = slope of $s = 4py_1/(y_1^2 = 4p^2)$. Now by using formula 7 of the Appendix to Chap. 3 we find that $\tan \alpha = 2p/y_1$ and $\tan \beta = 2p/y_1$. Hence $\alpha = \beta$.



CHAPTER 4, SECTION 5, SECOND SET

1. (a) $T\cos\theta = T_0 =$ constant. No.
   (b) $T\sin\theta = wx$. Yes.
2. $T = (wx/2y)\sqrt{x^2 + 4y^2}$.
3. The equation of the cable is of the form $y = x^2/4p$. The point $(60, 15)$ lies on the parabola. $w = {}^{150}/_{120} = {}^{5}/_{4}$ tons/ft. Then from Exercise 2, where we now calculate T at $x = 60$, $y = 15$, we have $T = 75\sqrt{5}$ tons.
4. Since $y = x^2/240$, we can substitute in the formula for T of Exercise 2. Then $T = (w/2)\sqrt{(240)^2 + 4x^2}$. We see that T increases with x and is a maximum at $x = \pm 60$ and a minimum at $x = 0$.
5. Yes as long as the weight of the roadway per horizontal foot (that is, not along the curved roadway) is constant.
6. We can use (17) but in place of wx we must use $5x^2$. Then $y' = 5x^2/T_0$ and $y = (5x^3/3T_0)+C$. We can determine C as in the text to be 0. However, this answer is correct only to the right of the origin. To the left the answer is $y = -5x^3/3T_0$ because x is negative. Or we may say that $y' = -5x^2/T_0$ because y' is negative and so obtain y for negative x.

CHAPTER 4, SECTION 6

1. Tangent exists everywhere except at $(0, 0)$.
2. No because there is no unique slope at $(0, 0)$.
3. The graph consists of the negative x-axis and the ray which starts at the origin and lies in the first quadrant and makes an angle of $45°$ with the positive x-axis. Then for $x < 0$, $y' = 0$ and for $x > 0$, $y' = 1$. There is no derivative at $(0, 0)$.

4. The slope of the chord or secant joining $(x_0-\Delta x,\ f(x_0-\Delta x))$ and $(x_0+\Delta x,\ f(x_0+\Delta x))$.
5. If we add and subtract $f(x_0)$ in the numerator we have

$$\frac{f(x_0 + \Delta x) - f(x_0) + f(x_0) - f(x_0\Delta x)}{2\Delta x}$$

and this equals

$$\tfrac{1}{2}\left[\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} + \frac{f(x_0) - f(x_0 - \Delta x)}{\Delta x}\right].$$

We can see geometrically that apart from the factor $\tfrac{1}{2}$ the first quotient is the slope of the secant joining $(x_0, f(x_0))$ and $(x_0 + x, f(x_0 + \Delta x))$ and the second quotient is the slope of the secant joining $(x_0 - \Delta x, f(x_0 - \Delta x))$ and $(x_0, f(x_0))$. Both slopes approach the slope of the tangent at $(x_0, f(x_0))$ as Δx approaches 0 and the entire quantity approaches $f'(x_0)$.

Analytically the first quotient is in the form of $\Delta y/\Delta x$ (see also Exercise 7) and surely approaches $f'(x_0)$. To write the second quotient in the customary form of $[f(x + \Delta x) - f(x)]/\Delta x$, let us first write $[f(x_0 - \Delta x) - f(x_0)]/(-\Delta x)$. Now in the definition of the derivative Δx can be negative. Let us use a new Δx which is the negative of the old Δx. Then our quotient becomes $[f(x_0 + \Delta x) - f(x_0)]/\Delta x$ and this approaches $f'(x_0)$, even though this Δx is negative, because $\Delta y/\Delta x$ must have the same limit when Δx approaches 0 through positive or negative values.

Solutions to Chapter 5

CHAPTER 5, SECTION 2, FIRST SET

1. (b) $y' = 1$; (c) $y' = 5x^4$; (d) $y' = 20x^4$; (e) $y' = 10x^9$; (g) $y' = 8x^7$; (h) $y' = (7/2)x^6$.
2. The limitation to a positive integral value of n is necessary because after step (8) we use the fact that there are (n-1) terms in the brackets. This is so because when n is a positive integer the binomial expansion in (5) has n+1 terms. If n were any other kind of number the binomial expansion would contain an infinite number of terms and the statement on line 4 of p. that $\Delta x(n-1)|A|$ approaches 0 as Δx does would not be possible.
3. Yes and because nx^{n-1} is 0 as it should be when $y = 1$.
4. Use the method in the text which leads to (9). Here $n = 3$.
5. $V = x^3$. Then $dV/dx = 3x^2$.
6. This is the derivative of $y = x^3$ at $x = a$. Hence $3a^2$.
7. We can think of p as x_0 and q as Δx. Then the answer in the text is obvious.
8. If we use the suggestion we get $2[f(x+t)-f(x)]/t$. Hence $2f'(x)$.

CHAPTER 5, SECTION 2, SECOND SET

1. (b) $y = x^4/4+C$; (c) $y = 3x^2/2+C$; (e) $y = x^6/6+C$; (f) $y = 7x^{11}/11+C$; (g) $y = f(x) = 2x^6/3+C$; (h) $y = f(x) = x^2+C$; (i) $y = f(x) = 4x^3/3+C$.
2. (b) $y = x^4/4+C$; (d) $y = x^2/2+C$; (e) $y = x+C$.

CHAPTER 5, SECTION 3

2. By repeating the derivation of (15) with a replacing 5 we obtain $x_2 = \frac{1}{3}[2x_1 + (a/x_1^2)]$.
3. By repeating the method used to derive (15) but with $y = x^2 - 5$ we obtain $x_2 = \frac{1}{2}[x_1 + (5/x_1)]$.
4. Repeat the work of Exercise 3 with a replacing 5. Then $x_2 = \frac{1}{2}[x_1 + (a/x_1)]$.
5. Repeat the method used to derive (15) but work with $y = x^n - a$. Then $x_2 = (1/n)[(n-1)x_1 + (a/x_1^{n-1})]$.
6. The method applies. One uses the method used to derive (15) but applied to $y = x^3 - 7x + 5$. One obtains $x_2 = (2x_1^3 - 5)/(3x_1^2 - 7)$. The root which is approximated depends on which root the choice of x_1 is close to.
7. No.
8. Let $y = f(x)$. Take a_1 to be an approximation to a root of $f(x)=0$ and let $y_1 = f(a_1)$. The equation of the tangent at a_1 is $y - y_1 = f'(a_1)(x-a_1)$. To obtain the point where the tangent cuts the x-axis, set $y = 0$ and solve for x. Then $x = a_2 = a_1 - f(a_1)/f'(a_1)$. Under proper theoretical conditions, namely, that $f(a_1)$ and $f''(a_1)$ have the same sign and $f''(x)$ does not change sign in $a_1 \le x \le r$, then a_2 is a better approximation to r than a_1.

CHAPTER 5, SECTION 4

1. (d) $y' = (5/3)x^{-2/3}$; (f) $y' = (1/4)x^{-3/4}$; (g) $y' = (p/3)x^{(p/3)-1}$; (i) $y' = (5/3)t^{2/3}$; (j) $\dot{s} = (3/2)t^{-1/2}$; (l) $\dot{s} = (1/3)t^{-2/3}$; (m) $f'(x) = (5/3)x^{2/3}$; (n) $f'(x) = (5/3)x^{2/3}+3$.
2. (c) $y = (15/4)x^{4/3}+C$; (d) $y = (3/5)x^{5/3}+C$; (e) $s = (9/5)t^{5/3}+C$; (f) $s = (3/2)t^{2/3}$ (i) $s = (5/8)t^{8/5}+C$; (k) $s = (8/3)t^{3/2}+C$; (l) $s = 3t^{1/3}+C$; (m) $f(x) = (3/8)x^{8/3}+(2/5)x^{5/2}+C$; (n) $f(x) = (4/3)x^{3/2}+2x^{1/2}+C$.
3. (b) $y = 2x^{1/2}+C$; (d) $s = 2t^{1/2}+C$; (f) $s = (5/8)t^{8/5}+C$.
4. No. The variable may be taken on the values .9, .99, .999, etc.
5. As x approaches 0, the values of $1/x^2$ become arbitrarily large.
6. $\Delta y/\Delta x = [1/(x_0+\Delta x) - 1/x_0]/\Delta x = -1/x_0(x_0+\Delta x)$. Hence $y' = -1/x_0^2$ for $x_0 \neq 0$.
7. As x increases from 0 to ∞, the slope decreases from ∞ to 0.
8. For the upper half of the parabola $y = \sqrt{8}\sqrt{x}$. Then $y' = (\sqrt{8}/2)x^{-1/2} = 2x^{-1/2}$. At (2,4) $y' = \sqrt{2}/\sqrt{2} = 1$. For the lower half of the parabola $y' = -\sqrt{2}^{-1/2}$. At $(2,-4)$, $y' = -1$.
9. For the upper half of the parabola $y = 4x^{1/2}$. $y' = 2x^{-1/2}$. At $x = 4$, $y' = 1$. Then the equation of the tangent is $(y-8) = 1(x-4)$. For the lower half of the parabola $y = -4x^{1/2}$ and $y' = -2x^{-1/2}$. At $x = 4$, $y' = -1$. The equation of the tangent is $(y+8) = -1(x-4)$.
10. See the answer to exercise 7.
11. No. $dv/ds = 4/\sqrt{s}$ and as s increases, dv/ds decreases.
12. $dT/d\ell = (2\pi/\sqrt{32})(1/2)\ell^{-1/2} = (\pi/4\sqrt{2})\ell^{-1/2}$. Yes.

Solutions to Chapter 6

CHAPTER 6, SECTION 2

1. (a) Continuous for all x; (b) continuous for all x; (c) continuous where defined, i.e., for $x \le 1$; (d) continuous except $x = 0$; (e) continuous where defined, i.e., for $x \ne 3$; (f) continuous where defined, i.e., for $x \ne 1$.
2. No; it jumps from 0° to 180° as P crosses the maximum point of the curve.
3. Yes, because the slope is at first positive, then 0 at the maximum point, and then negative.

CHAPTER 6, SECTION 3

1. (b) $y' = {}^{20}\!/_{3}x^{-1/3}$; (d) $y' = 20x^3 + {}^{7}\!/_{3}x^{-2/3}$;
   (f) $y' = 21x^2 + 14x$; (h) $y' = 3x^2 - 4x - 3$;
   (i) $y' = 4x^{-1/2}$; (j) $y' = 2x^{-1/2} + 2x^{-2/3}$;
   (l) $y' = x^{-1/2} + (\sqrt[3]{6}/3)x^{-2/3}$.
3. $y' = f'(x) - g'(x)$.
4. $y' = cf'(x)$.
5. (a) $y' = 30x - 3x^2$; for $x = 2$; $y' = 48$ ft/mile.
   (c) The slope of the graph is, of course, the value of y'.
   (d) Yes. By contrast in the vertical motions discussed in Chap. 3 the graph is not a picture of the motion.
6. The rate of change of the volume of a sphere at any value of the radius is the surface area of the sphere. Since $V = {}^{4}\!/_{3}\pi r^3$, $V' = 4\pi r^2$ and this is the value of the surface area.
7. $dC/dx = x^2 + 2$.
8. $p = (x-80)/10$. $R = xp = x^2/10 + 8x$. $dR/dx = (x/5) + 8$.
9. $dR/dx = 12x^2 - 6x$.
10. $P(x) = x^2 + 4x - x^3/3 - 2x - 4$. $dP/dx = -x^2 + 2x$.
11. $dC/dx = 3 + 4x$. When $x = 50$, $dC/dx = 203$. This is the cost of producing the 51st unit.

CHAPTER 6, SECTION 4

1. (a) Use (31); (b) Use (32); Ans. $y' = 4x^3 + 6x^2 - 2x - 2$; (c) Use (40);
   (d) Use (40); Ans. $y' = (x^2 + 6x + 1)/(x + 3)^2$; (e) Use (40);
   (f) Use (40) first, but to differentiate $x(x^2 - 1)$ in the numerator one must use (31); Ans. $y' = (2x^3 + 9x^2 - 3)/(x + 3)^2$;
   (g) Use (40);
   (h) Differentiate the sum and use (40) to differentiate the second term; Ans. $y' = 14x - 3/(x - 1)^2$.

3. $y' = [g(x)f'(x)-f(x)g'(x)]/g^2(x)$.
4. $f(x)g(x) = x^4(2x)+(2x+7)4x^3 = 2x^5+8x^4+28x^3$.
5. $y = \sqrt{x}/x^2+1)$. $y' = [(x^2+1)(1/2)x^{-1/2} - \sqrt{x}(2x)]/(x^2+1)^2 = [-(3/2)x^{3/2}+(1/2)x^{-1/2}]/(x^2+1)^2$.
6. $y' = [2x/(x+1)^2]-1/x^2$.
7. Just to use the product law $y' = (1/2)x^{-1/2}(1/x) - x^{1/2}(-1/x^2)$. This can be simplified to $(-1/2)(1/x^{3/2})$.
8. $C = (2500+3x+7x^2)/(100+2.5x^2) \cdot dC/dx = [(100+2.5x^2)(3+14x)-(2500+3x+7x^2)(5x)]/(100+2.5x^2)$. This can be simplified to $(-7.5x^2-6100x+300)/(100+2.5x^2)^2$.
9. Yes. The argument is the same.
10. Starting with the step $y = x^k = x \cdot x^{k-1}$ we may apply the result (31). Now $y' = x\,d(x^{k-1})/dx + x^{k-1} \cdot 1$. By the hypothesis of the mathematical induction process $y' = x(k-1)x^{k-2} + x^{k-1} = kx^{k-1}$. Since the result holds for $n = k$ on the assumption that it holds for $n = k-1$ and since it holds for $n = 1$, the result holds for all positive integral n.
11. $R = xp$. By using (31) we have $dR/dx = x(dp/dx)+p$.
12. $A = C(x)/x$. By using (40) we have $dA/dx = x(dC/dx)-C(x))/x^2$, dC/dx is M.
13. $V = 2500(1+t)^{-1}$; $dV/dt = -2500/(1+t)^2$. Clearly dV/dt is greatest when $t = 0$.
14. $R = xp = [640x/(x+9)]-40x$. $dR/dx = [5760/(x+9)^2]-40$. Now $dR/dx > 0$ when $[5760/(x+9)^2]-40 > 0$, or $5760 > 40(x+9)^2$, or $x < 3$.

CHAPTER 6, SECTION 5

1. (a) By Theorem 8 we may integrate x^2 and multiply the result by 8.
   (b) Same argument as in (a). Use (37) of Chap. 5 to integrate. Ans. $y = {}^{21}\!/_5 x^{5/3} + C$.
   (c) Write y' as $\sqrt{8}\sqrt{x} = \sqrt{8}x^{1/2}$ and use Theorem 8 and (37) of Chap. 5.
   (d) Use Theorem 9 and the results of (b) and (c)
   (e) Using Theorem 9, we may integrate each term separately
   (f) $y' = \sqrt{4}\sqrt{x} = 2x^{1/2}$. Then $y = {}^4\!/_3 x^{3/2} + C$.
   (g) Integrate each term separately. Ans. $y = -{}^7\!/_3 x^3 - 3x^2 + 3x + C$.
   (h) First divide through so that $y' = x^2 + 3$.
   (i) First divide through so that $y' = x^{7/2} + 3x^{3/2}$. Ans. $y = {}^2\!/_9 x^{9/2} + {}^6\!/_5 x^{5/2} + C$.
   (j) Since a, b and c are constants we may use Theorem 9.
   (k) Use Theorem 9 and integrate each term separately. Ans. $y = {}^3\!/_5 x^5 - {}^7\!/_3 x^3 + {}^5\!/_2 x^2 - 6x + C$.

2. (b) $y = 14x^{1/2}+C$; (c) $y = 7x+(8/3)x^3+C$;
   (e) $y = (x^3/3)-(21/5)x^{5/3}+C$; (f) $y = (2/3)x^{3/2}+(9/5)x^{5/3}+C$;
   (h) $y = t^3-2t^2+C$.

CHAPTER 6, SECTION 7

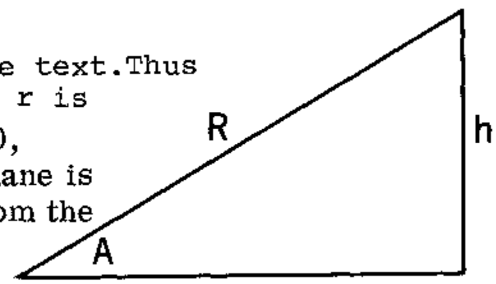
1. (a) Use (52); (b) Use (52), $y' = -\frac{3}{2}x^{-5/2}$;
   (c) We can write $y = 2x^{-1}$ and use (52) or differentiate as a quotient;
   (d) Write $y = x^{-1/2}$. Then by (52), $y' = -\frac{1}{2}x^{-3/2}$;
   (e) Write $y = (1/\sqrt{3})x^{-1/2}$;
   (f) Write $y = x^{-1} + 7x^{-2}$; then $y' = -x^{-2} - 14x^{-3}$. One can also differentiate as a quotient.
2. (a) Use (53). (b) Use (53). Ans. $y = 2x^{1/2} + C$.
   (c) $y' = x^{-1/2}$; now use (b). (d) $y' = \frac{1}{8}x^{-2}$; by (53) $y = -\frac{1}{8}x^{-1} + C$.
   (e) $y' = (1/\sqrt{8})x^{-1/2}$; then $y = (2/\sqrt{8})x^{1/2} + C$.
3. (b) $y = (-1/4)x^{-4} - 2x^{-1/2} + C$; (c) $y = -4x^{1/4} + C$;
   (d) $y = (-2/5)x^{-5/2} + C$; (f) $y = (-3/2)x^{-2} - 2x^{-1} + C$.
4. The tangent at any point x_0 is $y - 1/x_0 = (-1/x_0^2)(x - x_0)$. Hence in Figure 6-8 the coordinates of the points K, P, L are respectively $(0, 2/x_0)$, $(x_0, 1/x_0)$, $(2x_0, 0)$ from which the result follows at once.
5. Using Figure 6-8, the area cut off is $\frac{1}{2}\overline{OK}\cdot\overline{OL} = \frac{1}{2}(2/x_0)(2x_0) = 2 = \text{constant}$.
6. Since $I = A/4\pi r^2 = (A/4\pi)r^{-2}$, $I' = (-A/2\pi)r^{-3}$. Now let $r = 20$ and then $r = 200$ to obtain the text's answers. When r is very large, a small change in r causes very little change in the value of I because I varies inversely as the square of r or, speaking physically, I is spread out over such a large sphere that per unit area I is very small and changes very little as r changes.
7. $dC/dx = (1/2)x^{-1/2} - (1/2)x^{-3/2}$. When $x = 100$, $dC/dx = 1/20 - 1/2000$. Yes. Even under efficient production it must cost more to produce additional units of a commodity.

CHAPTER 6, SECTION 8

1. By (59), $W = GmM[(1/r) - (1/r_1)]$ where r is the final distance of the object from the center of the earth. Here $r = R$. Using (61) the result follows.
2. Use the formula of Exercise 1 with $m = 100$, $R = 4000 \times 5280$ and $r_1 = 4500 \times 5280$. Ans. $(22{,}528 \cdot 10^6)/3$ ft-pdl.
3. $8{,}448 \cdot 10^6$ ft-pdl. It is greater because for points above the surface the force of gravity is actually less than $32\,m$.
4. The work done in raising the satellite is numerically the same as the work done by gravity in pulling the satellite down. Hence use the formula in Exercise 1 with $m = 1000$ and $r_1 = 1500 \cdot 5280$ ft. The result is $18{,}432 \cdot 10^7$ ft-pdl.
5. By (58), $W = GmM/r + C$. Since $W = 0$ when $r = R$, $C = -GmM/R$ and $W = GmM/r - GmM/R$. Using (61), $W = 32\,mR[(R/r) - 1]$.
6. Measure r downward from the top of the well. Then the force on a length of cable of length r is the force (weight) per unit length times r. Thus $F = 32\,mr = 64r$. Using the relation $dW/dr = F$ derived in the text, we find

(since $W = 0$ for $r = 0$), $W = 32r^2$. Thus the work to lift the entire cable is $32(200)^2 = 1{,}280{,}000$ ft-pdl.

7. In addition to the work done in Exercise 6, we must lift 300 lbs a distance of 200 feet. Thus the additional work is $32(300) \times 200$ ft-pdl and so the total work is 3,200,000 ft-pdl.
8. a) yes; repeat the derivation of (59) verbatim.
   b) Repeat the derivation of (59) and Exercise 1 with 32 replaced by 5.3 and R being the radius of the moon. Ans. $W = 5.3\,mR(1 - R/r_1)$.
9. Use the answer to Exercise 8 with $m = 100$, $r_1 = 500 \cdot 5280$, and $R = 1100 \cdot 5280$. Ans. $96{,}195 \cdot 10^4$ ft-pdl.
10. Use the result of Exercise 8 with $m = 1000$, $r_1 = 1500 \cdot 5280$ and $R = 1100 \cdot 5280$.
11. a) Use the relation $dW/dr=F$ derived in the text. Thus $dW/dr=w\sin A$ and $W=(w \sin A)r+C$ where r is the distance the object is pushed. At $r = 0$, $W = 0$, hence $C = 0$. If the length of the plane is denoted by R, we have $W = wR \sin A$. From the figure $\sin A = h/R$, thus $W = wh$.



   b) Notice that this is the same result as would be obtained if the object were lifted straight up against gravity!

Solutions to Chapter 7

CHAPTER 7, SECTION 3

1. (a) Let $u = x^3 + 1$. Then $y = u^4$. Apply (12).
   (b) Let $u = x^2 - 7x + 6$. Then $y = u^4$ and by (12), $dy/dx = 4(x^2 - 7x + 6)^3(2x - 7)$.
   (c) Let $u = t^2-5$ and apply (12). Then $ds/dt = 8(t^2-5)t$.
   (d) Write as $(a^2-x^2)^{-2}$ and let $u = a^2-x^2$. Apply (12). Then $dy/dx = 4x(a^2-x^2)^{-3}$.
   (e) Differentiate as a quotient of two functions and in differentiating $5-2x$ let $u = 5-2x$. Likewise for $5+2x$. Ans. $10/(5-2x)^2$.
   (f) Let $u = x^2 + 1$. Then $y = u^{1/3}$. Apply (12).
   (g) Let $u = x/(x+1)$. Then $y = u^4$. Ans. $dy/dx = 4x^3/(x+1)^5$.
   (h) Let $u = x^2 - x$. Then $y = u^{1/2}$. Apply (12).
   (i) First differentiate as a product so that $dy/dx = \sqrt{x^2-x}\cdot 2x + (x^2+1)\,d(\sqrt{x^2-x})/dx$. To differentiate $\sqrt{x^2-x}$, let $u = x^2 - x$ and let $z = u^{1/2}$. Then calculate $dz/dx = (dz/du)(du/dx)$ and substitute the result in the expression for dy/dx. Ans. $dy/dx = (6x^3 - 5x^2 + 2x - 1)/2\sqrt{x^2-x}$.
   (j) Let $u = x/(1+x)$. Then $y = u^{1/2}$. Apply (12).
   (k) One can write $y = \sqrt{1-x^2}$. Let $u = 1 - x^2$. Then $y = u^{1/2}$. Apply (12). Ans. $dy/dx = -x/\sqrt{1-x^2}$.
   (l) Let $u = y'$. Then $y = u^2$. Then $dy/dx = 2y'y''$.

2. We are given w as a function of x and are told that x is a function of t. We want dw/dt. Now $dw/dt = (dw/dx)(dx/dt)$. We can calculate dw/dx from the given formula and we have that $dx/dt = 100$.

3. $R = x\sqrt{250-9x} = x(250-9x)^{1/2}$. $dR/dx = (250-9x)^{1/2} + x(1/2)(250-9x)^{-1/2}(-9)$.

4. (a) Differentiate first as a product. $y' = (x^2+2)^3(1) + (x-3)\,d(x^2+2)^3/dx$. Now let $u = x^2+2$ and apply (12). Then $y' = (x^2+2)^3 + (x-3)3(x^2+2)^2 2x$. One can simplify the result to get the text's answer.
   (b) Let $u = (x-1)/(x+1)$ so that $y = u^{1/2}$. Then by (12) $y' = (1/2)u^{-1/2}du/dx$. To find du/dx use the quotient rule. $du/dx = 2/(x+1)^2$.
   (c) Let $u = 5x^2+1$. Then by (12) $y' = (-2/3)(5x^2+1)^{-5/3}(10x)$.

CHAPTER 7, SECTION 4

1. (a) $x(dy/dx)+y = 0$; (b) $2x+2y(dy/dx) = 0$; (c) $6x+4y(dy/dx) = 0$; (d) $2x+x(dy/dx)+y+2y(dy/dx) = 0$, hence $dy/dx = (-2x-y)/(x+2y)$; (e) $3y^2(dy/dx)+x2y(dy/dx)+y^2+dy/dx+2 = 0$; (f) $2y(dy/dx) = 4$.
2. $2x+2y(dy/dx) = 0$. On the lower half of the circle we must take $y = -\sqrt{25-x^2}$.
3. $2y(dy/dx) = 8$. Then $dy/dx = 4/y$. At P, $x = 2$ and $y = -4$. Hence $dy/dx = -1$.
4. In differentiating $dy/dx = -x/y$ we must regard y as a function of x and differentiate the right side as a quotient. Hence $d^2y/dx^2 = -(y-xy')/y^2$. If we substitute the value of y' in this last equation we get the text's answer.
5. Here $dy/dx = 2p/y$ and at (x_0,y_0), $dy/dx = 2p/y_0$. Now use the point-slope form of the equation of the straight line.
6. If we set $y = 0$ in the answer to Exercise 5 and solve for x we obtain $x = x_0 - y_0^2/2p$. But $y_0^2 = 4px_0$.
7. The slope of the normal is the negative reciprocal of the slope of tangent. The latter (Exer. 5) is $2p/y_0$. Hence the equation of the normal is $y - y_0 = -(y_0/2p)(x - x_0)$. To obtain the x-intercept, set $y = 0$ and solve for x.

CHAPTER 7, SECTION 5

1. Take any point (x,y) on the circle and use the Pythagorean theorem.
2. Slope of $AC = y/(x+1)$. Slope of $BC = y/(x-1)$. But $y = \sqrt{1-x^2}$ so that the two slopes are the negative reciprocals of each other.
3. By comparison with (27) we see that $a = 4$ and $b = 3$. Since $c = \sqrt{a^2 - b^2}$, $c = \sqrt{7}$; $e = c/a = \sqrt{7}/4$.
4. Divide both sides by 80 so that $x^2/10 + y^2/8 = 1$. Now follow Exercise 1. Ans. $a = \sqrt{10}$, $b = \sqrt{8}$, $c = \sqrt{2}$, $e = \sqrt{5}/5$.
5. Since $a = 6$ and $b = \sqrt{10}$, $2a = 12$ and $2b = 2\sqrt{10}$.
6. (a) By (27) we get the text answer. (b) Here $2c = 4$ and since $b^2 = a^2 - c^2$, $b = \sqrt{60}$. Then $x^2/64 + y^2/60 = 1$.
   (b) We have by (27), $x^2/25b^2 + y^2/b^2 = 1$. Since $(7, 2)$ lies on the curve we may substitute 7 for x and 2 for y and $b^2 = {}^{149}/_{25}$. Hence $x^2/149 + 25y^2/149 = 1$.
   (d) $2a = 12$ and $2b = 8$. Now use (27).
7. We want twice the y-value at $x = c$. By (27) when $x = c$, $y^2 = (a^2 - c^2)(b^2/a^2) = b^4/a^2$. Then $2y = 2b^2/a$.
8. Repeat the derivation of (27) but with x and y interchanged.
9. We may use (27). Here $2a = 50$ and $b = 25$.
10. The least distance is $a - c$ and the greatest distance is $a + c$. Then $(a - c)/(a + c) = {}^{29}/_{30}$. Divide numerator and denominator by a. Then $(1 - e)/(1 + e) = {}^{29}/_{30}$. Solve for e.
11. If we compare with (37) we have $a = 4$ and $b = 3$. Since $c^2 = a^2 + b^2$, $c = 5$. $c/a = {}^5/_4$.
12. If we divide through by 80 we have $x^2/10 - y^2/8 = 1$. Now repeat Exercise 11 $a = \sqrt{10}$, $b = \sqrt{8}$, $c = \sqrt{18}$, $e = 3/\sqrt{5}$.
13. $2a = 12$ and $2b = 6$.
14. (a) Use (37). (b) $2c = 20$, so that $c = 10$. Since $b^2 = c^2 - a^2$, $b = 6$. Now use (37). (c) Use (37). (d) $2a = 12$ and $2b = 8$. Now use (37).
15. Since for the hyperbola $a^2 + b^2 = c^2$ and $a < c$, all we can conclude is that a may be greater than, equal to or less than b.
16. We want twice the y-value at $x = c$. By (37) at $x = c$, $y^2 = (c^2 - a^2)(b^2/a^2)$. Since $c^2 - a^2 = b^2$, $y = b^2/a$.
17. Repeat the derivation of (37) but with x and y interchanged.

18 . The argument given in connection with (38) rests on the fact that if a product <u>vanishes</u> then one of the factors must vanish. To apply this argument correctly in the present case we should consider the product $(5x + 2y - 25)(5x - 2y - 16)$ and conclude that the equation $25x^2 - 4y^2 - 205x + 18y + 400 = 0$ represents the two lines.

CHAPTER 7, SECTION 6

1. We find from $y = (b/a)\sqrt{x^2 - a^2}$ that $y' = (b/a)x/\sqrt{x^2 - a^2}$. At $x = a$ the slope is infinite. As x increases the slope decreases and as x becomes infinite the quantity $x/\sqrt{x^2 - a^2}$ approaches 1 because x is very large compared to a. Then the slope approaches b/a which is the slope of the asymptote $y = bx/a$.
2. From the given equation we have, by differentiating, $8x + 10yy' = 0$ so that $y' = -4x/5y$. At $x = 1$, $y = \pm 6/\sqrt{5}$ and $y' = \mp(2\sqrt{5}/15)$. Now use the point-slope form of the equation of the straight line.
3. At $x = -1$, $y = \pm 6/\sqrt{5}$. The slope (see Exer. 2) is $y' = \pm(2\sqrt{5}/15)$. Hence, using the point-slope form gives $y \mp 6\sqrt{5} = \pm(2\sqrt{5}/15)(x + 1)$.
4. The method is the same as in Exercise 2.
5. At $x = 5$, $y = \pm\sqrt{12}$. From the given equation $y' = 4x/5y$. At $x = 5$, $y' = \pm 2/\sqrt{3}$. Hence $y \mp \sqrt{12} = \pm(2/\sqrt{3})(x - 5)$.
6. The method is the same as in Exercise 5. Here $x = -5$.
7. (a) We shall show that the angle between FP and the tangent equals the angle between $F'P$ and the tangent by using formula (7) of the Appendix to Chap. 3. The tangent line has the slope $-b^2x/a^2y$. The line FP has the slope determined by (x, y) and $(c, 0)$ and so $y/(x - c)$. In using formula (7) we must remember to let m_2 be the slope of the line with the larger inclination and that our formula gives the tangent of the angle between the upward directions on the lines. Then application of (7) gives for the first angle $(-b^2x^2 - a^2y^2 + b^2xc)/(a^2xy - a^2yc - b^2xy)$. In the numerator we use the fact that $b^2x^2 + a^2y^2 = a^2b^2$ and in the denominator we use the fact that $a^2 - b^2 = c^2$. Then the fraction becomes b^2/yc. In using formula (7) to get the angle between $F'P$ and the tangent we must remember again that the formula gives the tangent of the angle between the upward directions of the two lines. This angle is the supplement of the one we are interested in and the tangent of the latter is the negative of the one we find by formula (7). Hence we apply (7) with $m_2 = -b^2x/a^2y$ and $m_1 = y/(x + c)$, as above, and take the negative of the result. This negative is also b^2/yc. Hence the two angles are equal.

   (b) The law of reflection for light says that the reflected ray will travel toward F.

8. The problem assumes that $a^2 > b^2$. As k varies from 0 to b^2, the locus is an ellipse. For $k = b^2$ there is no locus (though if we multiply through first by $(a^2 - k)(b^2 - k)$ we get $y = 0$, which is the x-axis). For k between b^2 and a^2 we have a hyperbola. For $k = a^2$ there is no locus (or, as before) the y-axis. For $k > a^2$ there is no locus.
9. The student could be asked to show that a confocal ellipse and hyperbola can be represented in the form given in Exercise 8. Thus if $x^2/a^2 + y^2/b^2 = 1$ is the equation of an ellipse and $x^2/A^2 - Y^2/B^2 = 1$ is the equation of a hyperbola we know that $b^2 = a^2 - c^2$ and $B^2 = c^2 - A^2$ if the two curves have the same foci. If we write $A^2 = a^2 - k$, which does not restrict A because k is still arbitrary, then $B^2 = c^2 - a^2 + k$. Thus the hyperbola becomes $X^2/(a^2 - k)$ $- Y^2/(c^2 - a^2 + k) = 1$ or, reverting to the usual x and y, $x^2/(a^2 - k)$ $- y^2/(k - b^2) = 1$ or $x^2/(a^2 - k) + y^2/(b^2 - k) = 1$. That is, a hyperbola confocal with an ellipse can be represented with an a^2 and b^2 which differ from those of the ellipse by the quantity k, except that for the hyperbola $a^2 > k > b^2$. Then the slope of the ellipse is $-b^2x/a^2y$ and the slope of the hyperbola is $x(k - b^2)/y(a^2 - k)$. The points of intersection of ellipse and hyperbola are obtained by solving the two equations simultaneously and they are given by $x^2 = a^2(a^2 - k)/(a^2 - b^2)$, $y^2 = b^2(k - b^2)/(a^2 - b^2)$. To make the algebra easier all we need show is that the product of the slopes of the ellipse and hyperbola, which product is $-b^2(k - b^2)x^2/a^2(a^2 - k)y^2$ has the value -1 at the points of intersection. If we substitute the x^2 and y^2 of the points of intersection we do obtain -1.
10. The tangent line to the ellipse at $(x_0 y_0)$ may be written as $b^2x_0x + a^2y_0y$ $= b^2x_0^2 + a^2y_0^2 = a^2b^2$. Thus the distance of any point (x, y) to the line is $(b^2x_0x + a^2y_0y - a^2b^2)/\sqrt{(b^2x_0)^2 + (a^2y_0)^2}$. The foci have coordinates (c, o), $(-c, o)$. Thus the product under consideration is given by $(a^2b^2 - b^2x_0c)(a^2b^2 + b^2x_0c)/(b^4x_0^2 + a^4y_0^2)$. Multiplying out, using $c^2 = a^2 - b^2$ in the numerator and $a^2(a^2y_0^2) = a^2(a^2b^2 - b^2x_0^2)$ in the denominator, this fraction reduces to b^2.
11. The method is precisely the same as in Exercise 7. One need watch only the differences in signs.
12. The slope at any point (x_0, y_0) of a hyperbola is $y' = b^2x_0/a^2y_0$. The equation of the tangent at that point is $y - y_0 = (b^2x_0/a^2y_0)(x - x_0)$ or $b^2x_0x - a^2y_0y$ $= b^2x_0^2 - a^2y_0^2$. The equations of the asymptotes are $bx - ay = 0$ and $bx + ay = 0$. We find the point of intersection of the tangent and the first asymptote and the same for the second. Then the distance from (x_0, y_0) to each point of intersection is the same.
13. The slope of the tangent line at (x_0, y_0) is $y' = 2p/y_0$. The equation of the tangent line is $y - y_0 = (2p/y_0)(x - x_0)$. Let $y = 0$ and solve for x. Then $x = x_0 - y_0^2/2p$. But $y_0^2/2p = 2x_0$.

14. Since the slope of the tangent to the ellipse at P which has coordinates (x_0, y_0), say, is $-b^2x_0/a^2y_0$, the slope of the normal is a^2y_0/b^2x_0 and the equation of the normal is $y - y_0 = (a^2y_0/b^2x_0)(x - x_0)$. The length OG is the x-intercept of the normal. Set $y = 0$ in the equation of the normal. Then $x = x_0 - (b^2/a^2)x_0 = x_0[1 - (b^2/a^2)] = x_0(c^2/a^2) = e^2x_0$.
15. The slope of the tangent to the ellipse at P, which has coordinates (x_0, y_0) say, is $-(b^2x_0/a^2y_0)$ and the equation of the tangent is $y - y_0 = -(b^2x_0/a^2y_0)(x - x_0)$. The length OT is the x-intercept of the tangent. Set $y = 0$ and solve for x. Then $x = a^2y_0^2/b^2x_0 + x_0$. This is OT. Since $ON = x_0$, $ON \cdot OT = a^2y_0^2/b^2 + x_0^2$. If we add and use the fact that $b^2x_0^2 + a^2y_0^2 = a^2b^2$, we have the result. The second part of the problem calls for the same procedure except that Ot is the y-intercept of the tangent and $OM = y_0$.

CHAPTER 7, SECTION 7

1. (a) Ignore temporarily the factor 4. Then if $u = (x^2 + 1)$ we have the form (47) and (48) gives the integral. The 4's cancel and the text's answer results.
   (b) Let $u = x^2 + 1$. Then we have the form (47), and (48) gives $y = (x^2 + 1)^4 + C$.
   (c) We can write the given dy/dx as $dy/dx = 1/2(x^2+1)^3 2x$. Now ignore the 1/2 for the moment and we have the form (47). Then (48) and the factor 1/2 yield the text's answer.
   (d) Let $u = (x^2+7x)$. Then apart from the constant factor 4 which can be temporarily ignored, our dy/dx is in the form (47). Ans. $y = (x^2+7x)^4+C$.
   (e) Same as (d) except for the factor 4.
   (f) If we let $u = x^2+6x$, then we need 2x+6 as our du/dx. Hence we write $dy/dx = 1/2(x^2+6x)^4(2x+6)$. Now except for the constant factor 1/2, dy/dx is in the form (47). Then by (48), $dy/dx = (x^2+6x)^5/10+C$.
   (g) If we write $dy/dx = (x^2+6)^{-5}2x$ and let $u = x^2+6$ we have the form (47). Then (48) gives the text's answer.
   (h) We could expand $(1+5x)^5$. But it is easier to let $u = 1+5x$. Then $du/dx = 5$. Hence write $dy/dx = 1/5(1+5x)^5 5$ and so except for the constant factor 1/5 we have the form (47). Then, by (48), $y = (1+5x)^6/30+C$.
   (i) $dy/dx = (1+5x)^{1/2}$. Now use the method of (h) to obtain the text's answer.
   (j) $dy/dx = (2-3x)^{-1/2}$. Let $u = 2-3x$. Then $du/dx = -3$. Hence write $dy/dx = -1/3(2-3x)^{-1/2}(-3)$. Then by (48) $y = -2/3(2-3x)^{1/2}+C$.
   (k) $dy/dx = (x^2+4)^{-3/2}x = 1/2(x^2+4)^{-3/2}2x$. Let $u = x^2+4$. Then apart from the constant factor 1/2 we have (47) and (48) gives the text's answer.
   (l) We can write $dy/dx = 2(1+x/2)^3(1/2)$. Now let $u = 1+x/2$. Then (48) yields $y = 1/2[1+(x/2)]^4+C$.
   (m) In this exercise the choice of u is not obviously helpful and must be regarded as a trial. If we do let $u = 4+5x$ then $dy/dx = \sqrt{u}(u-4)/5 = (1/5)u^{3/2}-(4/5)u^{1/2}$. This expression is still

not in the form (47) because each term lacks the factor du/dx or 5 and so we write $dy/dx = (1/25)u^{3/2}5-(4/25)u^{1/2}5$. Now each term, apart from a constant factor, is in the form (47) and (48) applied to each term, yields the text's answer.

2. (a) Let $u = x^2+5$. Then we have the form (48).
   (b) Write the given integral as $(1/2)\int(x^2+5)^2 2x\,dx$. Let $u = x^2+5$. Then we have the form (48) and the answer is $(1/2)(x^2+5)^3/3 = (1/6)(x^2+5)^3+C$.
   (c) Write the given integral as $\int(x^2+5)^{-2}x\,dx$ and apply the method of (b). The answer is $(-1/2)(x^2+5)^{-1}+C$.
   (d) Write the given integral as $(3/2)\int(x^2+5)^{-2}2x\,dx$ and proceed as in (c). The answer is $(-3/2)(x^2+5)^{-1}+C$.
   (e) Let $u = x^3+x^2$. Then the integral is in the form (48).
   (f) Let $u = x^2+7x$. Then the integral is in the form (48). The answer is $2(x^2+7x)^{1/2}+C$.
   (g) Write the given integral as $1/2\int(x^2+2x)^{-2}(2x+2)dx$ and let $u = (x^2+2x)$. By (48) the answer is $(-1/2)(x^2+2x)^{-1}+C$.
   (h) Write the given integral as $(3/2)\int(x+2x)^{-2}(2x+2)dx$ and use (48) with $u = x+2x$. The answer is $(-3/2)(x^2+2x^{-1}+C$.
3. Using the suggestion in the text we get $dW/dx = k(c-x)/[p^2+(c-x)^2]^{3/2}$. Then $dW/dx = k[p^2+(c-x)^2]^{-3/2}(c-x)$. Let $u = p^2+(c-x)^2$. Then $du/dx = -2(c-x)$. Then $dW/dx = -(k/2)u^{-3/2}du/dx$. By (47) $W = ku^{-1/2}+C = k[p^2+(c-x)^2]^{-1/2}+C$. Now x varies from 0 to ℓ. When $x = 0$, $W = 0$. Hence $C = -k(p^2+c^2)^{-1/2}$. When $x = \ell$, $W = k[p^2+(c-\ell)^2]^{-1/2}-k(p^2+c^2)^{-1/2}$. From Fig. 7-20 we see that $W = k/b-k/a$.
4. If we expand $(2x-1)^4/8$ we get $2x^4-4x^3+3x^2-x+\frac{1}{8}+C$ for the integral. On the other hand if we expand $(2x-1)^3$ we get $8x^3-12x^2+6x-1$ and if we integrate we get $2x^4-4x^3+3x^2-x+C$. The two results seem to differ by $\frac{1}{8}$. But the constant of integration in the second case can be taken to be $\frac{1}{8}+C'$ where C' is some new constant. That is, the two solutions differ seemingly by $\frac{1}{8}$ but the constant of integration can always be adjusted to take into account any constant, such as the $\frac{1}{8}$ here, because the constant of integration is an arbitrary value.
5. If we use the fact that $f(x) = g'(x)$ then $y' = g(x)g'(x)$. Now let $u = g(x)$. Then $du/dx = g'(x)$. Then $y' = u(du/dx)$ and we have the form (47). By (48), $y = (u^2/2) + C = \{[g(x)]^2/2\} + C$.

CHAPTER 7, SECTION 8

1. We have but to apply (65) in which $r_1 = 5000 \cdot 5280$, $GM = 32R^2$, and $R = 4000 \cdot 5280$. The rest is just arithmetic to get the text's answer.
2. Formula (65) gives the velocity acquired in falling from rest or zero velocity and from the height r_1 above the center of the earth to the surface. As the text points out, if we shoot an object up from the surface with the velocity acquired in falling to the surface it will arrive at height r_1 (above the center of the earth) with 0 velocity. Then the least velocity with which the object should be shot up is given by (65) with $R = 4000 \cdot 5280$, $GM = 32R^2$ $r_1 = 8000 \cdot 5280$. The arithmetic gives $3200\sqrt{66}$ ft/sec.

3. We could solve this problem by the methods of Chapter 3, Section 3. See for example Exercise 12 there. Or we can use the technique of this chapter to argue that $\dot{v} = -32$ and, by (61), $v(dv/dr) = -32$ so that $v^2/2 = -32r + C$. If we measure r from the surface of the earth (which we may do in this problem as opposed to the use of (59)) then what we want is that v should be 0 when $r = 4000 \cdot 5280$. Then $C = 32(4000 \cdot 5280)$ and $v^2/2 = -32r + 32(4000 \cdot 5280)$. We seek the value of v when $r = 0$. This is $6400\sqrt{33}$ ft/sec. This value is larger than the value in Exercise 2, as it should be, because the acceleration of -32 is larger than the true acceleration of gravity and so more initial velocity is required to have the object reach a height of 4000 miles.
4. For the particle which falls under the true acceleration of gravity, the ve-velocity on reaching the earth's surface is given by (65) where $r_1 = 2R$. Then $v_0^2/2 = GM/2R$ or $v_0^2 = 32R$ in view of (58). To obtain the velocity acquired by a particle falling from rest with the acceleration of 32 ft/sec² we may use the reasoning of Exercise 3 which leads to $v^2/2 = -32r + C$. Now $v = 0$ when $r = R/2$ so that $C = 16R$ and $v^2/2 = -32r + 16R$. When the object reaches the surface $r = 0$ and $v_0^2 = 32R$. Hence the two velocities are equal.
5. According to (68) the escape velocity is $8\sqrt{R}$. In Exercise 3 we calculated the velocity required to send an object 4000 miles up if the acceleration of gravity were 32 ft/sec² all the way. There we obtained $6400\sqrt{33}$. This is the value of $8\sqrt{R}$, since $R = 4000 \cdot 5280$.
6. The problem assumes the body has a velocity V in the downward direction. We may use the result (63) but must now determine C by the condition that at $r = r_1$, $v = V$. Then $V^2/2 = GM/r_1 + C$ or $v^2/2 = GM/r - GM/r_1 + V^2/2$. Since in the determination of C only V^2 enters we cannot be sure that the signs are correct. However $GM/r - GM/r_1$ is positive because $r < r_1$. Then the value of this difference adds to V^2, as it should to produce a larger v. When $r = R$ we obtain $v = -\sqrt{V^2 + 2GM/R - 2GM/r_1}$, the minus entering because a downward v should be negative.
7. The velocity of 1000 ft/sec. is in the upward (positive) direction. The initial velocity must be such that it supplies the loss in v which is due to gravity and still leave a velocity of 1000 at the height of 8000 miles from the center. Hence the velocity we seek is really the opposite sign from that calculated in the latter part of Exercise 6. That is, we seek $v_0 = +\sqrt{V^2 + 2GM/R - 2GM/r_1}$ where $V = 1000$, $R = 4000 \cdot 5280$ and $r_1 = 8000 \cdot 5280$. The calculation yields 26,016 ft/sec.
8. Use (65) with $v_0 = 10{,}000$, and $R = 4000 \cdot 5280$ and solve for r_1. The answer is 320 miles approximately.
9. Again use (65) with $v_0 = 5280$ and $R = 4000 \cdot 5280$ and solve for r_1. The answer is about 84 miles.
10. The object will certainly never return. In mathematical terms $\lim_{r_1 \to \infty} GM/r_1 > 0$.
11. In the theory of the section we have but to let M stand for the mass of the moon and R stand for the radius of the moon.
12. In view of the answer to Exercise 11, we can use (67) if we let $M = M/81$ and R be $4R/15$. Then $v_0 = \sqrt{(2GM/81)(15/4R)} = \sqrt{(15/324)(2GM/R)}$. Now M

and R still refer to the mass and radius of the earth. Hence $v_0 = \sqrt{15/324}\,\sqrt{2GM/R} = .21\sqrt{2GM/R}$. We can use (68), so that $v_0 = .21(7\text{mi/sec}) = 1.50$ mi/sec approx.

13. We use (65) with $r_1 = (240{,}000)(5280)$ and $R = (4000)(5280)$ to calculate v_0. The arithmetic yields $8\sqrt{R}\,\sqrt{59/60}$. Since $8\sqrt{R} = 7$ mi/sec, the result is about 6.88 mi/sec.
14. Here we use (65) but with M, the mass of the moon, R, the radius of the moon, and $r_1 = 240{,}000$ mi. From the data of Exercise (12) we know that $M = M/81$ and $R = 4R/15$. Then (65) reads: $v_0^2/2 = (GM/81)(15/4R) - (GM/81)(1/240{,}000 \cdot 5280)$. We can see by looking at the value of v_0 in Exercise 12, that the value here will be slightly less than the value there. The result is 1.49 mi/sec approximately.
15. We start with (63) and let $v = v_0$ when $r = R$. Then $C = (v_0^2/2) - (GM/R)$. Substituting this value for C in (63) gives the text's answer.
16. (a) The given differential equation in the Suggestion replaces (59) and the reason is that two accelerations both in the direction of negative r act on the object and so the accelerations add. To integrate we replace d^2r/dt^2 by $v(dv/dr)$, as justified in (61), and integrate with respect to r. Then $v^2/2 = GM/r + GS/(r+d) + C$. We want v to be 0 when $r = r_1$. This gives the value of C so that (1) $v^2/2 = GM(1/r - 1/r_1) + GS[1/(r+d) - 1/(r_1+d)]$. This is the velocity acquired in falling from rest at the distance r_1 from the surface of the earth to the distance r from the surface. The velocity acquired on reaching the surface of the earth is obtained by replacing r by R. This latter velocity is also the velocity with which the object must be shot up from the surface of the earth to just reach r_1.

    (b) As in the derivation of (67) we use (1) above in which $r = R$ and r_1 becomes infinite. Then the escape velocity v_0 is given by $v_0^2/2 = GM/R + GS/(R+d)$. To calculate v_0 we use the fact that $S = 330{,}000M$ and $d = 93 \cdot 10^6$ mi. Then $v_0^2 = (2GM/R) + [2G \cdot 330{,}000M/(R + 93 \cdot 10^6)]$, or $v_0^2 = (2GM/R)\{1 + 330{,}000/[1 + (93 \cdot 10^6/R)]\} = (2GM/R)(15.2)$. The arithmetic gives, since $\sqrt{2GM/R} = 7$, $v_0 = 27.3$ mi/sec.
17. The equation involving r_s says that the earth's gravitational attraction on the mass m just equals the moon's gravitational attraction of the mass m. Then, as noted in the text, $r_s = 54R$. Hence we must now calculate the velocity required to shoot an object up from the surface of the earth to just reach the point $r_1 = 54R$. We use (65) so that $v_0^2 = 2GM/R - 2GM/54R = (2GM/R)(1 - 1/54)$. Then $v_0 = \sqrt{2GM/R}\sqrt{53/54} = 7(.99) = 6.93$ mi/sec. The result in the text, 6.86, is .99 times the more accurate value than the 7 mi/sec, namely, .99(6.93).
18. The object must be shot up from the moon to just reach the point which is 6R from the moon's surface, R being the radius of the earth. We use (65) with $M = M/81$ and $R = 4R/15$ and $r_1 = 6R$. Then $v_0^2 = 2G(M/81)(15/4R) - 2G(M/81)(1/6R)$. Or $v_0^2 = (2GM/R)(15/432 - 1/486) = (2GM/R)(.038)$. Then $v_0 = 1.40$ mi/sec. The precise answer depends, of course, on the accuracy to which the calculations are carried.

19. (a) If r is measured positively in the direction up from the earth's surface toward the moon, then the acceleration due to the earth's attraction is negative and is $-GM/r^2$. The moon's attraction is in the positive direction and when the object is r units from the earth's center it is $60R - r$ units from the moon's center because the distance between centers is 60R. Then the acceleration due to the moon is $G(M/81)/(60R - r)^2$. These two accelerations act at all distances r from the earth's surface to the moon's surface. Hence the differential equation in the text takes into account the continuing accelerations of the earth and moon.

(b) The differential equation of the text can be written as $v(dv/dr) = -GM/r^2 + (GM/81)/(60R-r)^2$. Integration yields $v^2/2 = GM/r - \int (GM/81)(60R-r)^{-2}(-1)dr + C$ or $v^2/2 = GM/r + (GM/81)/(60R-r) + C$. We now require that $v = 0$ at the stagnation point where $r = 54R$. This yields $C = -8GM/9.54R$, so that $v^2/2 = GM/r + GM/81(60R-r) - 8GM/9.54R$. We now wish to find v when $r = R$. Hence $v^2/2 = (GM/R)(1+(1/81.59)-(8/9.54))$. If one uses $GM = 32R^2$, then v is in ft/sec. First one obtains $v^2/2 = (GM/R)(.9837)$. Now depending on the accuracy of the calculation one obtains $v = 6.96$ mi/sec.

CHAPTER 7, SECTION 9

1. Since $V = e^3$, $dV/dt = 3e^2 de/dt$. Now $de/dt = 0.05$ and thus the text answer.
2. The volume of a cone is $V = \pi r^2 h/3$. We are given that $r = h/3$ so that $V = \pi h^3/27$. Then $dV/dt = (\pi h^2/9)dh/dt$. We are given that $dV/dt = 1728.3$ cu.in/min. when $h = 6$. Then $dh/dt = 1296/\pi$ in/min.
3. The suggestion is to assume that $dV/dt = kA$. Since $V = 4\pi r^3/3$, $dV/dt = 4\pi r^2 dr/dt$. Also $A = 4\pi r^2$. Then $dr/dt = k$ and $r = kt + C$. When $t = 0$, $r = 1/2$. Then $r = kt + 1/2$. When $t = 6$, $r = 1/4$ so that $k = -1/24$. Then $r = (-t/24) + 1/2$.
4. We may use the text result of sect. 8, namely $dW/dt = -(GmM/r^2)dr/dt$. At the surface of the earth $r = R$ and $GM/R^2 = 32$; $m = 10$, and $dr/dt = 25{,}000$. Then $dW/dt = 8\cdot 10^6$. At the height of 100 miles, $r = 4100$. Just to compare results write $r = (4100/R)R$ where $R = 4000$ miles. Then substituting in dW/dt gives $(40/41)^2$ times the preceding result.
5. (a) From $A = \pi r^2$, we have $dA/dt = 2\pi r\, dr/dt$.
   (b) Here $r = 5$ and $dr/dt = 1/2$ so that $dA/dt = 5\pi$ sq ft/sec.
   (c) No. r certainly changes from instant to instant and dr/dt may also change from one instant to another.
   (d) No, because even if dr/dt is constant, r changes during the next second.
6. Since $A = s^2$, $dA/dt = 2s\, ds/dt$. Let $s = 5$ and $ds/dt = 1$. Then $dA/dt = 10$ sq ft/min.
7. The distance the first ship sails east is 20t, where t is measured from noon. The distance the second ship sails south in t hours is $25(t-1)$. The distance r between them is given by the Pythagorean theorem; $r = \sqrt{(20t)^2 + 25^2(t-1)^2} = \sqrt{1025t^2 - 1250t + 625}$. Then $dr/dt = (1025t - 625)/r^{1/2}$. Now let $t = 2$. Then $dr/dt = 285/\sqrt{89}$ ft/sec.

8. The volume of water at any r and h is $V = \pi r^2 h/3$. At any value of the water height h and the radius r of the water surface $r/h = 7/12$ by similar triangles. Then $V = 49\pi h^3/432$ and $dV/dt = \frac{49}{144}\pi h^2\, dh/dt$. The net change in dV/dt is 10. Hence when $h = 6$, $dh/dt = 40/49\pi$ ft/min.
9. From $y^2 = 2x$ we have $y' = 1/y = 1/\sqrt{2x}$. Think of y' as some new variable z. Then $z = (2x)^{-1/2}$ and $dz/dt = -\frac{1}{2}(2x)^{-3/2}[d(2x)/dx](dx/dt) = -(1/\sqrt{8x^3})(dx/dt)$. Now let $x = 32$ and $dx/dt = 1/2$. Then $dz/dt = -\frac{1}{1024}$ unit/min.
10. (a) The distance r from the station is $r = \sqrt{x^2 \; y^2}$. Since $y^2 = 4x$, $r = \sqrt{x^2 + 4x}$. Then $dr/dt = (dr/dx)(dx/dt) = [(x+2)/\sqrt{x^2+4x}]dx/dt$. At $x = 9$ and $dx/dt = 2$, $dr/dt = 22/\sqrt{117}$ unit/min.
    (b) When x is large the length r is nearly horizontal and increases at about the same rate as x does.

CHAPTER 7, APPENDIX, SECTION A2

1. (a) We use equations (6) on p. 186 with $\theta = 45°$. Then $x = (\sqrt{2}/2)(x' - y')$ and $y = (\sqrt{2}/2)(x' + y')$. Substitution in $x^2 + 4xy + y^2 = 16$ gives the text answer.
    (b) If $\tan\theta = 2$ then $\sin\theta = 2/\sqrt{5}$ and $\cos\theta = 1/\sqrt{5}$. Now use (6) on p. 186. Then $x = (1/\sqrt{5})(x' - 2y')$ and $y = (1/\sqrt{5})(2x' + y')$. We substitute these values in $3x^2 - 4xy + 10 = 0$. Ans. $x'^2 - 4y'^2 = 10$.
    (c) If $\tan\theta = 3$ then $\sin\theta = 3/\sqrt{10}$ and $\cos\theta = 1/\sqrt{10}$. Now proceed as in parts (a) and (b). $x = (1/\sqrt{10})(x' - 3y')$; $y = (1/\sqrt{10})(3x' + y')$. Substitution in $3x^2 - 3xy - y^2 = 5$ yields $3x'^2 - 7y'^2 = 10$.
    (d) As in (b), $x = (1/\sqrt{5})(x' - 2y')$ and $y = (1/\sqrt{5})(2x' + y')$. Substitution in the given equation yields $x'^2 = 0$.
2. Yes because the points of the circle have the same position with respect to the new axes that they do with respect to the old.
3. (a) To determine the angle of rotation we use (10) on p. 187. In this exercise $A = 1$, $B = 2\sqrt{3}$ and $C = -1$. Then $\tan 2\theta = \sqrt{3}$. In this case we may recognize at once or see from the trigonometric tables that $2\theta = 60°$ and so $\theta = 30°$. Then $\sin\theta = \frac{1}{2}$ and $\cos\theta = \sqrt{3}/2$. We now use the equations (6) on p. Then $x = \frac{1}{2}(\sqrt{3}x' - y')$ and $y = \frac{1}{2}(x' + \sqrt{3}y')$. Substitution in the given equation yields the text answer.
    (b) The angle of rotation is given by $\tan 2\theta = -16/(17 - 17)$. This means, as pointed out in the text, that $2\theta = 90°$ and $\theta = 45°$. Then $\sin\theta = \sqrt{2}/2$ and $\cos\theta = \sqrt{2}/2$. From (6) on p. 186, $x = (\sqrt{2}/2)(x' - y')$ and $y = (\sqrt{2}/2)(x' + y')$. Substitution in the given equation yields $9x'^2 + 25y'^2 = 225$.
    (c) The angle of rotation is given by $\tan 2\theta = -\sqrt{3}$. Since $\tan(180 - 2\theta) = -\tan 2\theta$, $\tan(180 - 2\theta) = \sqrt{3}$ and $180 - 2\theta = 60°$. Then $2\theta = 120$ and $\theta = 60°$. Then $\sin\theta = \sqrt{3}/2$ and $\cos\theta = \frac{1}{2}$. By (6) of p.186, $x = \frac{1}{2}(\sqrt{3}x' - y')$ and $y = \frac{1}{2}(x' + \sqrt{3}y')$. Substitution in the given equation yields the text's answer.
    (d) $\operatorname{Tan} 2\theta = 4/(1 - 1)$. This means $2\theta = 90°$ and $\theta = 45°$. Then, as in (b), $x = (\sqrt{2}/2)(x' - y')$ and $y = (\sqrt{2}/2)(x' + y')$. Substitution in the given equation yields $3x'^2 - y'^2 = 16$.

(e) $\operatorname{Tan} 2\theta = -2/(1-1)$. Then $2\theta = 90$, and $\theta = 45°$. By (6) of p.186 , $x = (\sqrt{2}/2)(x' - y')$ and $y = (\sqrt{2}/2)(x' + y')$. Substitution in the given equation yields the text's answer.

(f) $\operatorname{Tan} 2\theta = 1/(0-0)$ or $2\theta = 90°$ and $\theta = 45°$. The equations (6) are as in (e) and substitution in $xy = 12$ yields $x'^2 - y'^2 = 24$.

(g) $\operatorname{Tan} 2\theta = 14/(25-25)$. Then $2\theta = 90°$ and $\theta = 45°$. The equations (6) are as in (e) and substitution in the given equation yields the text's answer.

4. Yes. If we took an x'-axis parallel to the straight line we could put its equation in the form $y' = d$.

CHAPTER 7, APPENDIX, SECTION A3

1. (a) If we let $x = x' + h$ and $y = y' + k$, substitute in the given equation, take the terms involving x' and set the coefficient equal to 0, we find $h = -16$. Likewise if we set the coefficient of $y' = 0$ we find $k = 2$. The sum of the constant terms in the new equation becomes -212. Hence the answer in the text. Graph the new equation with respect to the new axes and then put in the (x, y)-axes.

(b) The method is the same as in (a). We find that $h = 2$ and $k = 3/4$. The transformed equation is $3x'^2 + 4y'^2 - 29/4 = 0$. Graph as recommended in (a).

(c) Use the method of (a). We find that $h = -5$ and $k = 3$. The final equation is $x'^2 + 4y'^2 - 11 = 0$.

(d) If we replace x by $x' + h$ and y by $y' + k$ we can set the coefficient of $x' = 0$ and find that $h = -3$. Now set the sum of the constant terms equal to 0; this yields $k = -23/4$. The resulting equation is that in the text.

(e) The method is the same as in (d). We find that $h = 4$ and $k = -77/4$. The resulting equation is in the text.

(f) The method is the same as in (e) except that now we first find k to eliminate the y' term and then set the constant equal to 0 to determine h. We find that $h = -1$ and $k = 5$. The resulting equation is $y'^2 - 6x' = 0$.

(g) The method is as in (a). We find that $h = -4$ and $k = -7$. The resulting equation is as in the text.

(h) The method is as in (a). We find that $h = 3$ and $k = -2$. The resulting equation is $x'^2 + y'^2 = 25$.

(i) If we replace x by $x' + h$ and y by $y' + k$ we find that we can take k to be 0 and so eliminate the y' term. Setting the constant term equal to 0 given $h = 2$. The resulting equation is in the text.

(j) The method is as in (f). We find that $h = -1$ and $k = 3$. The resulting equation is $y'^2 = 8x'$.

2. Yes. By choosing new axes with origin at any point on the line we can eliminate the constant term. This is obvious geometrically because the line would go through the new origin.

3. We see by inspection that if we let $x' = x - 3$ and $y' = y + 2$ the equation reduces to the standard form. Then the equations for translation are $x = x' + 3$ and $y = y' - 2$.
4. (a) If we substitute $x = x' + h$ and $y = y' + k$ in the given equation we find that we can eliminate the x' term by setting its coefficient equal to 0. This gives $h = -b/2a$. Now we can fix k so that the constant term is 0. This gives $k = ah^2 + bh + c$, and in view of the value of h, $k = (4ac - b^2)/4a$. The new equation becomes $y' = ax'^2$ which is of the form $y = x^2/4p$ with $a = 1/4p$.
   (b) The focus of $y = x^2/4p$ is at $(0, p)$ or for the equation $y' = ax'^2$ at $(0, 1/4a)$. The translation determined in (a) is $x = x' - b/2a$, $y = y' + (4ac - b^2)/4a$. The coordinates of the focus with respect to the new axes are $(0, 1/4a)$. These are the x' and y' of the focus. Then the x and y of the focus are $(-b/2a, (4ac - b^2 + 1)/4a)$. The coordinates of the vertex of $y' = ax'^2$ are $(0, 0)$. Then the coordinates of the vertex in the xy-system are $(-b/2a, (4ac - b^2)/4a)$.
   (c) The directrix of $y = x^2/4p$ is $y = -p$. Then for $y' = ax'^2$ the directrix, since $p = 1/4a$, is $y' = -1/4a$. To obtain the equation of this line in the xy-system we have that $y = y' + k = (4ac - b^2 - 1)/4a$.
5. In each of the parts of Exercise 5, we could follow the method used in Exercise 1 of replacing x by $x' + h$ and y by $y' + k$ to eliminate the x and y terms. However it is well to teach the method of completing the square which in this exercise and the next one also gives the answers we want more readily.
   (a) Write the given equation as $16(x^2 - 8x) - 25(y^2 + 6y) = 369$. Now complete the square in each parenthesis and compensate by adding the equivalent term on the right side. Thus $16(x^2 - 8x + 16) - 25(y^2 + 6y + 9) = 369 + 256 - 225$ or $16(x - 4)^2 - 25(y + 3)^2 = 400$. Then if we let $x' = x - 4$ and $y' = y + 3$ we have $16x'^2 - 25y'^2 = 400$ or $x'^2/25 - y'^2/16 = 1$. We see that $a = 5$ and $b = 4$. The x' and y' of the center is $(0, 0)$. Hence the x and y of the center are $(4, -3)$.
   (b) Use the method in (a). The equation becomes $(x - 4)^2/36 - (y + 5)^2/36 = -1$ and the translated equation is $x'^2/36 - y'^2/36 = -1$. According to Exercise 17 on p.164 , $a = 6$ and $b = 6$. Since $x' = x - 4$ and $y = y' + 5$ the coordinates x and y of the center are $(4, -5)$.
   (c) Use the method of (a). After completing the square the equation becomes $(x + 3)^2/16 - (y - 2)^2/36 = 1$. The answers are in the text.
   (d) The numerical answers in this exercise would be simpler if the constant on the right side were changed to 288. Then the method of (a) leads to $(x + 6)^2/45 - (y - 4)^2/80 = 1$ and to $x'^2/45 - y'^2/80 = 1$. Then $a = \sqrt{45}$, $b = \sqrt{80}$ and the xy-center is $(-6, 4)$.
   (e) Use the method in (a). The given equation can be put in the form $(x + 4)^2/16 - (y - 5)^2/25 = -1$. According to Exercise 17 on p.164 , $a = 5$, $b = 4$ and the center is $(-4, 5)$.
6. See the introduction to Exercise 5.
   (a) Completing the square yields $(x - 3)^2/25 + (y + 6)^2/36 = 1$. We note here that the larger number of 25 and 36 is under the y^2 term. This means

that (see Exercise 8 on p. 163) that the foci are on y' -axis. Then $a = 6$, $b = 5$ and the center is $(3,-6)$.

(b) Completing the square yields $(x-3)^2/25 + (y-2)^2/16 = 1$. Hence $a = 5$, $b = 4$ and the center is $(3,2)$.

(c) Completing the square yields $(x+2)^2/25 + (y+3)^2/9 = 1$. Hence the results in the text.

(d) Completing the square yields $(x+3)^2/36 + (y-3)^2/9 = 1$. Hence $a = 6$, $b = 3$ and the center is $(-3,3)$.

(e) The exercise is $49x^2 + 16y^2 - 196x - 96y - 444 = 0$. Completing the square yields $(x-2)^2/16 + (y-3)^2/49 = 1$. Here as in (a) the foci are on the (new) y' -axis. Hence $a = 7$, $b = 4$ and the center is $(2,3)$.

7. (a) We rotate first. $\operatorname{Tan} 2\theta = -4/3$. To avoid decimals we shall use the formulas on the bottom of p.189 . We find that $\cos 2\theta = -3/5$, the minus entering because 2θ is a second quadrant angle. Then $\sin\theta = 2/\sqrt{5}$ and $\cos\theta = 1/\sqrt{5}$. The equations for rotation are $x = (1/\sqrt{5})(x' - 2y')$ and $y = (1/\sqrt{5})(2x' + y')$. Substitution in the given equation gives the final result. No translation of axes need be applied.

(b) $\operatorname{Tan} 2\theta = 6/(5-5)$. Hence $\theta = 45°$. The equations for rotation are $x = (\sqrt{2}/2)(x' - y')$ and $y = (\sqrt{2}/2)(x' + y')$. Substitution in the given equation yields $8x'^2 + 2y'^2 + 8\sqrt{2}\,x' - 14\sqrt{2}\,y' + 21 = 0$. Now translation of axes gives the text result.

(c) $\operatorname{Tan} 2\theta = 2/3$. To avoid decimals use the formulas on the bottom of p.189 . $\cos 2\theta = 3/\sqrt{13}$. Then $\sin\theta = \sqrt{(\sqrt{13}-3)/2\sqrt{13}}$ and $\cos\theta = \sqrt{(\sqrt{13}+3)/2\sqrt{13}}$. Also $\sin\theta\cos\theta = 1/\sqrt{13}$. Substitution of the equations (6) for rotation yields $x'^2(5/2 - \sqrt{13}/2) + y'^2(5/2 + \sqrt{13}/2) - 4 = 0$. No further simplification by translation can be obtained.

(d) $\operatorname{Tan} 2\theta = 2/3$. Hence the equations for rotation are the same as in (c). The result of the substitution gives $(2 + 11/\sqrt{13})x'^2 + (2 - 11/\sqrt{13})y'^2$ + linear terms. Translation of axes must now be applied to eliminate the linear terms.

(e) $\operatorname{Tan} 2\theta = -2/(1-1)$. Hence $\theta = 45°$ and we proceed as in (b). The result is in the text.

(f) $\operatorname{Tan} 2\theta = 2/(1-1)$. Hence $\theta = 45°$. Proceed as in (b). We obtain just by rotation $x'^2 - 2\sqrt{2}\,y' = 0$.

(g) Since there is no xy-term we need apply only translation. We can determine h to eliminate the x-term and then determine k to eliminate the constant. $h = -3$, $k = -1$.

(h) $\operatorname{Tan} 2\theta = 10/(13-13)$. Hence $\theta = 45°$. Proceed as in (b). The result of the rotation is $18x'^2 - 8y'^2 - 18x' + 24y' - 27 = 0$. Now translate axes.

CHAPTER 7, APPENDIX, SECTION A4

1. We have but to apply $B^2 - 4AC$ to each part of Exercise 7 of the preceding list. Thus in (a), $B = 4$, $A = 1$ and $C = 4$. Then $B^2 - 4AC = 0$ and the curve is a parabola. This fact is confirmed by the answer in (a).

2. The equation represents a degenerate hyperbola. That is, the graph consisting of two intersecting straight lines must be classed among the hyperbolas if we are to include all second degree equations among the conic sections.
3. (a) No. Slope of a line is defined relative to the x-axis. If we rotate the x-axis the slope must change.
   (b) If we substitute equations (6) in $y = mx + b$ we have $x' \sin\theta + y' \cos\theta = m(x' \cos\theta - y' \sin\theta) + b$. If we now write this equation in the form $y' = m'x' + b'$ we obtain the slope m' relative to the new axis.
4. (a) Yes. The angle is a geometric fact about the two lines and so is independent of the choice of axes.
   (b) If we have two lines $y = m_1x + b_1$ and $y = m_2x + b_2$ and rotate axes, according to Exercise 3(b), $m_1' = (m_1 \cos\theta - \sin\theta)/(\cos\theta + m_1 \sin\theta)$ and $m_2' = (m_2 \cos\theta - \sin\theta)/(\cos\theta + m_2 \sin\theta)$. If we now calculate $\tan\theta = (m_2' - m_1')/(1 + m_1'm_2')$ we obtain $\tan\theta = (m_2 - m_1)/(1 + m_1m_2)$.
5. No. The slope of a curve is defined to be the slope of the tangent line. The latter (see Exercise 3(a)) is not invariant under rotation.
6. (a) Yes. Under translation the x' -axis is parallel to the x-axis and so the inclination of the line and therefore its slope remains the same.
   (b) Replace y and x in $y = mx + b$ by $y' + k$ and $x' + h$, respectively. Then determine the slope in the transformed equation.
7. Yes. The reason is the same as in Exercise 4(a).
8. (a) Yes, because for any (x, y) the expression $x^2 + y^2$ represents the square of its distance from the origin. Under rotation this distance remains the same.
   (b) Replace x and y by the values given by (6). We find that $x^2 + y^2 = x'^2 + y'^2$.
9. (a) No, because under translation the distance of the point from the origin changes.
   (b) Replace x and y by $x' + h$ and $y' + k$, respectively. Then $x^2 + y^2 \neq x'^2 + y'^2$.

Solutions to Chapter 8

CHAPTER 8, SECTION 2

1. (a) $y' = 6x^2 - 18x - 24 = 6(x+1)(x-4)$. We see that $y' = 0$ at $x = 4$ and $x = -1$. For x slightly less than 4, y' is negative and for $x > 4$, y' is positive. Hence at $x = 4$, y has a minimum of -124. For $x < -1$, y' is positive and for x slightly greater than -1, say, 0, y' is negative. Hence y has a maximum at $x = -1$ and the y-value is 1.
   (b) The method is the same as in (a). Here $y' = -3(x+3)(x-1)$. Then y' is 0 at $x = 1$ and $x = -3$. At $x = 1$ there is a maximum of 20 and at $x = -3$ there is a minimum of -12.
   (c) The method is the same as in (a). $y' = 4x^3 - 4x = 4x(x-1)(x+1)$. At $x = 0$, y' changes from + to − ; hence at $x = 0$ there is a maximum which is 0. At $x = -1$, y' changes from − to + ; hence at $x = -1$, there is a minimum which is -1. Likewise at $x = 1$, there is a minimum of -1.
   (d) $y' = 2x - 2x^{-1}$. $y' = 0$ at $x = -1$. At $x = -1$, y' changes from − to +. Hence there is a minimum whose value is 2. At $x = 1$, y' changes from − to +. Hence there is a minimum whose value is 2.
   (e) $y' = -6(x-1)(x+1)/(x^2+1)^2$. $y' = 0$ when $x = -1$ and when $x = +1$. At $x = -1$, y' changes from − to +; hence there is a minimum whose value is -3. At $x = 1$, y' changes from + to −; hence there is a maximum whose value is 3.

CHAPTER 8, SECTION 3

1. (a) $y = 16-3x-9x^2$; $y' = -3-18x$, $y' = 0$ when $x = -1/6$. For x slightly less than $-1/6$, y' is negative and for x slightly greater, y' is positive. Hence at $x = -1/6$, y has a relative minimum. Substitute $x = -1/6$ in the function to find the minimum y-value.
   (b) $y = -8x+2$. Here $y' = -8$ and is never 0. Hence no relative maxima or minima.
   (c) $y = 2x^3-6x$; $y' = 6x^2-6$; $y' = 0$ when $x = +1$ and $x = -1$. For x slightly less than 1, y' is negative and for x slightly greater, y' is positive. Hence at $x = 1$ the function has a relative minimum. For x slightly less than -1, say $-5/4$, y' is positive and for x slightly greater than -1, say $-3/4$, y' is negative. Hence y has a relative maximum at $x = -1$.
   (d) $y = 2x^3+8x+3$; $y' = 6x^2+8$. y' is not 0 for any real values of x. Hence no relative maxima or minima.
   (e) $y = x^3$; $y' = 3x^2$. $y' = 0$ when $x = 0$. However, y' does not change sign at $x = o$. Hence no relative maxima or minima.
   (f) $y = x^4-2x^2+12$; $y' = 4x^3-4x = 4x(x-1)(x+1)$. Hence $x = 0$, -1 and 1 are possible values. By testing each in turn for a change in sign in y' at 0, -1, and 1 we see that at $x = 0$, there is

a relative maximum; at $x = -1$ there is a relative maximum; at $x = 1$ there is a relative maximum.

(g) $f(x) = x^4$; $f'(x) = 4x^3$. Hence $x = 0$ is a possible value. $f'(x)$ does change from negative to positive around $x = 0$. Hence at $x = 0$ there is a relative minimum.

(h) $f(x) = x + 1/x$; $f'(x) = 1 - 1/x^2$. Hence $x = 1$ and $x = -1$ are possible values. For x slightly less than -1, say $-5/4$, $f'(x)$ is positive and for x slightly more than -1, say $-3/4$, $f'(x)$ is negative. Hence there is a relative maximum at $x = -1$. At $x = 1$, $f'(x)$ changes from negative to positive; hence there is a relative minimum at $x = 1$.

(i) $f(x) = x\sqrt{x-1}$; $f'(x) = (x/2\sqrt{x-1}) + \sqrt{x-1}$. Now $f'(x) = 0$ at $x = 2/3$. But the function has no real value at $x = 2/3$. Hence there are no maxima and minima.

(j) $f(x) = x^2/(x-1)$; $f'(x) = (x^2-2x)/(x-1)^2$. Possible values are $x = 0$ and $x = 2$. At $x = 0$, $f'(x)$ changes from positive to negative. Hence there is a relative maximum there. At $x = 2$, $f'(x)$ changes from negative to positive. Hence there is a relative minimum there.

2. (a) $y' = -2x + 6$. Hence $x = 3$ is a possible relative maximum or minimum. At $x = 3$, y' changes from + to −. Hence there is a relative maximum whose value is 16. The absolute maxima and minima may occur at the end values 0 and 5. At $x = 0$, $y = 7$ and at $x = 5$, $y = 12$. Hence the relative maximum of 16 is also the absolute maximum and the absolute minimum is 7.

(b) $y' = 3x(x-2)$. At $x = 0$, y' changes from + to −. Hence there is a relative maximum of 4. At $x = 2$, y' changes from − to +. Hence there is a relative minimum of 0. At $x = -2$, $y' = -16$. This is the absolute minimum. At $x = 4$, $y = 20$. This is the absolute maximum.

3. $y' = 2(x-1)(x+1)^2 + 2(x+1)(x-1)^2 = 4x(x+1)(x-1)$. At $x = -1$, y' changes from − to +. Hence there is a relative minimum and its value is 0. At $x = 0$, y' changes from + to −. Hence there is a relative maximum whose value is 1. At $x = 1$, the behavior of y' is as at $x = -1$ and the relative minimum is again 0.

4. Here $y' = -2/3(x-1)^{1/3}$. This y' is never 0. However the function may have a relative maximum or minimum where the derivative fails to exist. We see that $(x-1)^{2/3}$ is positive for every value of x and this is subtracted from 3. The least we can subtract is 0 and this occurs when $x = 1$. Then $y = 3$ is a relative maximum. As x increases or decreases from the value of 1, y continually decreases.

5. (a) The function $y = x$ is an example. However $y = -x^2$ in the interval from $-\infty$ to 0 is a better example. Here as x increases, $f'(x)$ actually de-

creases. In fact in an interval to the left of any relative maximum we have another example.
(b) The latter two examples in (a) answer this point.
(c) Consider an interval to the left of a point where $f(x)$ has a minimum. There $f'(x)$ is increasing but $f(x)$ is decreasing.

6. No. The example of $y = x^3$ at $x = 0$ shows that y increases there but $y' = 0$.

CHAPTER 8, SECTION 4

1. (a) $y' = 3x^2 - 6x - 9 = 3(x-3)(x+1)$. The possible maxima and minima occur at $x = 3$ and $x = -1$. $y'' = 6x - 6$. At $x = -1$, y'' is negative. Hence a maximum occurs at $x = -1$ and this maximum is 7. At $x = 3$, y'' is positive. Hence a minimum occurs at $x = 3$ and this minimum is -25.
(b) $y' = 2x - 16x^{-2}$. $y' = 0$ at $x = 2$. $y'' = 2 + 32x^{-3}$. At $x = 2$ y'' is positive. Hence there is a minimum at $x = 2$ and $y = 12$.
(c) $y' = 2(x+1)(x-2)(2x-1)$. Hence there are possible maxima or minima at $x = -1$, $x = \frac{1}{2}$ and $x = 2$. $y'' = 4(x+1)(x-2) + 2(x+1)(2x-1) + 2(x-2)(2x-1)$. At $x = -1$, y'' is positive; hence a minimum which is 0. At $x = \frac{1}{2}$, y'' is negative; hence a maximum which is $\frac{81}{16}$. At $x = 2$, y'' is positive; hence a minimum which is 0.
(d) $y' = 4x^3 - 6x^2 - 4x = 2x(2x+1)(x-2)$. $y'' = 12x^2 - 12x - 4$. At $x = 0$, y'' is negative. Hence a maximum which is 1. At $x = -\frac{1}{2}$, y'' is positive. Hence a minimum which is $\frac{13}{16}$. At $x = 2$, y'' is positive; hence a minimum which is -7.
(e) $y' = (-ax^2 + a^3)/(x^2 + a)^2$. $y'' = (2ax^3 - 4a^3x)/(x^2 + a^2)^3$. At $x = -a$, y'' is positive; hence a minimum which is $-\frac{1}{2}$. At $x = a$, y'' is negative; hence a maximum which is $\frac{1}{2}$.
(f) $y' = 12x^3 - 12x^2 - 72x = 12x(x-3)(x+2)$. $y'' = 36x^2 - 24x - 72$. At $x = -2$, y'' is positive. Hence a minimum which is -4. At $x = 0$, y'' is negative; hence a maximum which is 60. At $x = 3$, y'' is positive; hence a minimum which is -129.
(g) $y' = 4x^3 - 4x = 4x(x-1)(x+1)$. $y'' = 12x^2 - 4$. At $x = -1$, y'' is positive; hence a minimum which is 9. At $x = 0$, y'' is negative; hence a maximum which is 10. At $x = 1$, y'' is positive; hence a minimum which is 9.
(h) $y' = 3(x^2 + 2x + 3)$. There are no real roots.

2. $y' = -3x^2 + 6x + 9$. Call y', z and find the maximum value of z. $z' = -6x + 6$. $z' = 0$ for $x = 1$. $z'' = -6$. Hence $x = 1$ is a maximum for z and this maximum is 12.

CHAPTER 8, SECTION 5

1. $\dot{h} = 80 - 32t$; $\ddot{h} = -32$. Hence at $t = 2\frac{1}{2}$ there is a maximum which is 100.
2. If x and y are the dimensions of any rectangle $P = 2x + 2y$. We also have the condition that $xy = A$ where A is constant. Then $y = A/x$ and $P = 2(x + A/x)$. $P' = 0$ implies $x = \pm\sqrt{A}$, $x = \sqrt{A}$ give the minimum. Thus for least perimeter $x = \sqrt{A}$ and $y = A/x = \sqrt{A}$; that is, we have the square.
3. $A = xy$ and $2x + y = 100$. Hence $A = 100x - 2x^2$ and $A' = 100 - 4x$. Then $x = 25$ and $y = 50$.
4. Let the distance from the foot of the altitude to B be denoted by t. Then $CB = \sqrt{h^2 + t^2}$, $CA = \sqrt{h^2 + (2a - t)^2}$ and $P = 2a + \sqrt{h^2 + t^2} + \sqrt{h^2 + (2a - t)^2}$. $P' = 0$ implies $t = a$ from which $CB = CA$ follows.
5. Let a be the given base and set $b = 2s - a - c$. Then $A = \sqrt{s(s - a)(a + c - s)(s - c)}$ where only c is variable. $dA/dc = 0$ implies $2s - 2c - a = 0$. Since $2s = a + b + c$, we have $b = c$.
6. Let x and y be the lengths of the sides. Then $P = 3x + 2y$. Since $xy = 864$, $y = 864/x$ and $P' = 3x + 1728/x$. Then $P' = 3 - 1728x^{-2}$. $P' = 0$ implies $x = 24$. Then $y = 36$.
7. If the sides have length x and y, then $\sqrt{x^2 + y^2} = 10$. The area $A = x\sqrt{100 - x^2}$. Then $A' = \sqrt{100 - x^2} - x^2/\sqrt{100 - x^2}$. $A' = 0$ implies $x = 5\sqrt{2}$. Then from $\sqrt{x^2 + y^2} = 10$ we have $y = 5\sqrt{2}$.
8. The surface area $A = 2\pi r^2 + 2\pi rh$. But $\pi r^2 h = 1$. Then $h = 1/\pi r^2$ and $A = 2\pi r^2 + 2/r$. $A' = 0$ implies the answers in the text.
9. We wish to minimize $D = \sqrt{x^2 + y^2}$ with $xy = A$, where A is constant. To save work we can argue that to minimize D is also to minimize D^2. Call D^2, z. Then $z = x^2 + y^2$ and $y = A/x$, so that $z = x^2 + A^2/x^2$. Hence $z' = 2x - 2A^2/x^{-3}$ and $z' = 0$ implies $x = \sqrt{A}$. Hence $y = \sqrt{A}$.
10. Since the rectangles are inscribed in a circle, the diameter is 2. We may then use the method of Exercise 7 with 10 replaced by 2. The result is a square with side $\sqrt{2}$.
11. Let (x, y) be any point on the parabola. Then the distance in question is $D = \sqrt{(x - p)^2 + y^2} + \sqrt{(x - a)^2 + (y - b)^2}$. To make D a function of one variable we can use the fact that $y^2 = 4px$ and we can replace x by its value in terms of y or y by its value in terms of x. Alternatively we can keep x and y and regard y as a function of x. However in the first radical we can replace y^2 by $4px$ and obtain $D = p + x + \sqrt{(x - a)^2 + (y - b)^2}$. Then $dD/dx = 1 + [(x - a) + (y - b)y']/\sqrt{(x - a)^2 + (y - b)^2}$. Set $dD/dx = 0$, replace y' by $2p/y$ and solve for y. To solve put the term 1 on the other side of the equation and then square both sides. One finds that $y - b$ is a factor of the equation and so $y = b$ is a root. Alternatively, one knows that $y = b$ should be a value of y at which $dD/dx = 0$. If we substitute b for y in dD/dx we get $dD/dx = 1 + (x - a)/\sqrt{(x - a)^2}$. Now $\sqrt{(x - a)^2} = |x - a|$ because the radical sign stands for a positive value. However the numerator $(x - a)$ is negative because x necessarily lies to the left of a. Hence $dD/dx = 1 + (-1) = 0$. One could test dD/dx to see that it changes sign at $y = b$ but this step is often omitted in physical problems and would be lengthy in this one.

If we calculate dF/dr we see that it is never zero.
For the given function $y' = 1 - x^2$. Then $y' = 0$ when $x = -1$ and $x = +1$. Since $y'' = 2x^{-3}$ we see that at $x = -1$, y'' is negative. Hence at $x = -1$ there is a maximum whose value is 0. At $x = 1$, y'' is positive and so there is a minimum there. The minimum value is 2 and larger than the maximum.

14. The assertion is false. Consider $y = x - a$
15. $y' = 0$ implies $x = -b/2a$. Since $y'' = 2a$, the value $-b/2a$ yields a maximum if $a < 0$.
16. $y' = 3x^2 + 2bx + 3$. We wish to have no real roots. The discriminant of this quadratic is $4b^2 - 36$ and we wish this to be negative. Then $|b|$ must be less than 3.
17. Since $y' = 3x^2$, y' is 0 at $x = 0$. However, y'' is also 0 and so we have no text. If we use the test that y' change sign at $x = 0$ we see that it does not. Moreover since y' is positive to the left and right of $x = 0$, the function is always increasing. Hence there is no relative maximum. However because the function is always increasing there is an absolute maximum at $x = 4$.
18. The rectangle will have dimensions $100 + x$ and y. The perimeter of the entire rectangle will be $200 + 2x + 2y = 300$. Then $A = (100 + x)y$ with $y = 50 - x$. Hence $A = (100 + x)(50 - x) = 5000 - 50x - x^2$. Then $dA/dx = -50 - 2x$ and there is no positive value of x which maximizes A. The domain of possible values for x is $0 \leq x \leq 50$. In this domain the maximum value of A is given by $x = 0$ because A decreases as x increases from 0 on. Then $x = 0$ and $y = 50$. The dimensions are 50 by 100.
19. If (x, y) is any point on the parabola then $D = \sqrt{(x-1)^2 + y^2}$. Since $y^2 = 4x$, by eliminating y we have $D = \sqrt{(x-1)^2 + 4x}$. As noted in Exercise 9 we can minimize D^2 in place of D. Let $z = D^2$. Then $z = (x-1)^2 + 4x$ and $z' = 2(x-1) + 4$. When $z' = 0$, $x = -1$. However $x = -1$ does not correspond to any point on the parabola and so there is no relative minimum for x-values of points on the parabola. However there may be an absolute minimum. The domain of x is $x \geq 0$. We see from the expression for z that $z = (x+1)^2$ and the smallest value occurs at $x = 0$.

    If we eliminate x from D we have $D = \sqrt{[(y^2/4) - 1]^2 + y^2}$ and $z = [(y^2/4) - 1]^2 + y^2$. Then $z' = [(y^2/4) - 1]y + 2y$ and $z' = 0$ when $y = 0$. Since $z'' = 3y^2/4 + 1$ we see that z'' is positive when $y = 0$. Hence $y = 0$ furnishes a minimum. Then $x = 0$.
20. Here $D = \sqrt{(x-c)^2 + y^2}$. As in Exercises 9 and 19 let us work with D^2 or z. Since $y^2 = (b^2/a^2)(a^2 - x^2)$, $z = (x-c)^2 + (b^2/a^2)(a^2 - x^2)$. Then $z' = 2(x-c) - 2(b^2/a^2)x$. Since $a^2 - b^2 = c^2$, $z' = 0$ when $x = a^2/c$. Since $a/c > 1$, this value of x is larger than a and so lies outside the domain of admissible x. However z can be put in the form $(cx - a^2)^2/a^2$. This expression is least when cx is as close to a^2 as possible and this is when x is a for x in th domain $-a$ to a.

    If we eliminate y from D^2 after writing $D^2 = x^2 - 2cx + c^2 + y^2$ by $x = a\sqrt{b^2 - y^2}/b$ and differentiate D^2 we find that the only real root of z is $y = 0$. This does furnish the relative minimum though the testing b z'' is lengthy (and can be ignored).

21. If we let $AC = x$, then the time for the trip as $t = \sqrt{1+x^2}/3 + (1-x)/5$. $dt/dx = x/3\sqrt{1+x^2} - {}^1/_5$. If we set this expression equal to 0 and solve for x we obtain $x = {}^3/_4$. We could test d^2t/dx^2 to see that $x = {}^3/_4$ furnishes a minimum.
22. In place of the previous dt/dx we have $dt/dx = x/4\sqrt{1+x^2} - 1/5$. If we set $dt/dx = 0$ and solve for x, we obtain $x = {}^4/_3$. Since x cannot be larger than 1, this value of x does not furnish a minimum in the domain $0 \leq x \leq 1$. However the man does save time by rowing as much as possible even to reach a point ${}^4/_3$ of a mile from A. Hence he should certainly save as much time as possible by rowing to B. Alternatively, we can see that dt/dx is negative for $0 \leq x \leq 1$, because $x/\sqrt{1+x^2} < 1/\sqrt{2}$ for $0 \leq x \leq 1$. Hence $x/4\sqrt{1+x^2} < 1/4\sqrt{2}$ which is less than ${}^1/_5$. Hence t decreases and is least when $x = 1$.
23. It is obvious in this case that using the diagonal saves as much time as possible since he can row as fast as he can walk. Hence the diagonal path PB is best.
24. Let $PA + PB + PC = S$. Then $S = 2\sqrt{x^2+6^2} + 3 - x$. Since $dS/dx = 2x/\sqrt{x^2+6^2} - 1$, $dS/dx = 0$ at $x = \sqrt{12}$. This value of x does not yield a point on CD. There is no relative minimum for P on CD. However dS/dx is negative for $0 \leq x \leq 3$ and so S decreases as x increases. Hence the minimum S occurs at $x = 3$ or when P is at C.
25. There is no relative maximum or minimum. The maximum and minimum values occur at the end values $x = r$ and $x = -r$ of the permissible x-values.
26. Denote the sum by S, then $S = (m_1 - m)^2 + (m_2 - m)^2 + \ldots + (m_n - m)^2$. Then $dS/dm = 0$ implies $(m - m_1) + (m - m_2) + \ldots + (m - m_n) = 0$ or $m = = (m_1 + m_2 + \ldots + m_n)/n$.
27. Follow the method in the text used to derive (16). Here B lies on the same side of CD as A does and $v_1 = v_2$. Hence we end up with $\sin\alpha/\sin\beta = V_1/V_1$ and so $\alpha = \beta$.
28. In Exercise 27 we prove that the time AP + PB is least but as the suggestion points out this also means least distance. Hence the problem is the same as Exercise 27 with just a different physical interpretation.
29. No matter where the bridge is placed the distance PQ must be covered in any case. Hence the problem reduces to minimizing AP + QB. Label the foot of the perpendicular from A to CD, R and the foot of the perpendicular from B to EF, S. The distance RS is fixed; let it be d. Let x denote RP. Then $QS = d - x$. Let AR be a and let BS be b. Then $AP + QB = \ell = \sqrt{a^2+x^2} + \sqrt{b^2+(d-x)^2}$. To minimize ℓ, we calculate $d\ell/dx$ which is $x/\sqrt{a^2+x^2} - (d-x)/\sqrt{b^2+(d-x)^2}$. This derivative set equal to 0 gives $x/\sqrt{a^2+x^2} = (d-x)/\sqrt{b^2+(d-x)^2}$. The left side $= \cos APR$ and the right side $= \cos QSB$. Then $\angle APR = \angle BQS$ and AP and QB are parallel.
30. Find ds/dt and set it equal to 0. There $t = 4.09$°Centigrade. d^2s/dt^2 is negative in $0 < t < 0$. Hence at $t = 4.09$ C the specific weight is a maximum.

CHAPTER 8, SECTION 6, FIRST SET

1. $A = \frac{C}{x} = x^2-6x+15$. Then $\frac{dA}{dx} = 2x-6 = 0$ so that $x = 3$ for minimum A. Hence the minimum A is $3^2-6.3+15$ or 6. Now $dC/dx = 3x^2-12x+15$. At $x = 3$, $dC/dx = 6$.
2. Profit = R-C and R = Px. Hence $P = 55x-3x^2-x^3+15x^2-76x-10 = -x^3+12x^2-21x-10$. Then $dP/dx = -3x^2+24x-21$. When $dP/dx = 0$, $x = 1$ and $x = 7$. At $x = 1$, $P = -20$ and this would mean a deficit. When $x = 7$, $P = 88$. If we test d^2P/dx^2 at $x = 7$, we find that it is negative so that $x = 7$ yields a maximum profit.
3. $P = R-C$. Hence $dP/dx = dR/dx-dC/dx$. When $dP/dx = 0$, $dR/dx = dC/dx$.
4. $P = px-C = 75x-2x^2 - 350-12-x^2/4 = (-9/4)x^2+63x-350$. $dP/dx = (-9/2)x+63$. When $dP/dx = 0$, $x = 14$. The price at maximum profit is $75-2\cdot14 = 47$. The maximum profit which occurs when $x = 14$ is $(-9/4)(14)^2+63\cdot14-350 = 92$ dollars.
5. $P = x(100-.10x)-1000-50x = -1000+50x-.10x^2$. Then $dP/dx = 50-.20x$. When $dP/dx = 0$, $x = 250$.
6. Let x be the number of articles to be produced. Then the total cost of x articles is $C = 100+x/2+x^2/100$. The average cost is $A = C/x = 100/x+\frac{1}{2}+x/100$. $dA/dx = -100/x^2+1/100$. When $dA/dx = 0$, $x^2 = 10,000$ or $x = 100$.
7. $C = a+c\sqrt{x}$. Then $A = C/x = a/x+c/\sqrt{x}$. $dA/dx = -a/x^2-c/2x^{3/2}$. Setting $dA/dx = 0$ gives $-a/x^2-c/2x^{3/2} = 0$. Multiply through by $x^{3/2}$. Then $-a/x^{1/2}-c = 0$ or $-a-cx^{1/2} = 0$. Hence $x^{1/2} = -a/c$ or $x = a^2/c^2$.
8. Let x be the interest rate offered. The dollars D attracted will be $D = kx$. The return to the bank will be $R = .07D$. The interest paid by the bank will be $xD = kx^2$. The profit is $P = .07kx-kx^2$. $dP/dx = .07k-2kx$. Hence when $dP/dx = 0$, $x = .035$.
9. Let the dimensions of the floor be x and y, with x the front and back lengths. Then if the cost per foot of the side walls and back is d, $C = 2dy+dx+2dx$. But $xy = 10,000$. Hence $y = 10,000/x$ and $C = 20,000d/x+3dx$. $dC/dx = -20,000d/x^2+3d$. When $dC/dx = 0$, $3x^2 = 20,000$ and $x = \sqrt{20,000/3}$. The height is immaterial.
10. The cost per mile times the miles per hour = cost per hour. At speed v the cost per hour is $C = 125+(1/10)v^3$. At speed v the miles per hour is v. Hence the cost per mile or $M = c/v = 125/v+(1/10)v^2$ Then $dM/dv = -125/v^2+(2/10)v$. When $dM/dv = 0$, $v = \sqrt[3]{625} = 5\sqrt[3]{5}$.
11. Let x be the number of new wells produced. Then the number of wells will be 25+x. The number of barrels of oil produced per well will be 100-3x. The total number of barrels T is $T = (25+x)(100-3x) = 2500+25x-3x^2$. $dT/dx = 25-6x$. Hence when $dT/dx = 0$, $x = 25/6$.
12. Let x be the number of cards to be printed. The cost $C = 10,000+5x+x^2$ (in pennies). The income is 100x. Hence the profit $P = 100x-10,000-5x-x^2$. $dP/dx = 100-5-2x$. When $dP/dx = 0$, $x = 47\frac{1}{2}$.
13. (a) Suppose the building contains x floors. Then the cost is $C = 500000 + 100000 + 200000 + \cdots + 100000x$. Then $C = 500000 + 100000(1+2+\cdots+x)$ or $C = 500000 + 100000[\frac{x}{2}(1+x)]$ or $C = 500000 + 50000x^2 + 50000x$. The profit P after the first year is $P = 300000x - 500000 - 50000x^2 - 50000x$. We find dP/dx and x when $dP/dx = 0$, $x = 2\frac{1}{2}$. Practically one would choose 2 or 3.
    (b) To maximize the percentage of profit we wish to maximize the return each year divided by the total cost. Then, using f(x) for the percentage,

$$f(x) = \frac{300000x}{500000+50000x^2+50000x} = \frac{30x}{50+5x+5x^2}$$

Using the quotient rule gives

$$f'(x) = \frac{1500-150x^2}{(50+5x+5x^2)^2}$$

Then $f'(x) = 0$ gives $x = \sqrt{10}$. Again practically x would be 3 or 4.

14. At x miles per hour the number of hours required to make the 100 mile trip is 100/x. The cost per hour is $C = 10+x^2/50$. Hence the cost of the trip is

$$C = \frac{100}{x}(10 + \frac{x^2}{50}) = \frac{1000}{x} + 2x.$$

Then $dC/dx = -1000/x^2+2$. At $dC/dx = 0$, $x = \sqrt{500}$.

15. This exercise and the next one follow the second illustrative example. Let x be the sale price. Then the reduction in price will be 30-x. For each $2 in 30-x there will be ten more sales. Hence the additional sales are $10(\frac{30-x}{2})$. The revenue will be $R = 500x + (10/2)(30-x)x$ or $R = 500x+150x-5x^2 = 250x-5x^2$. Then $dR/dx = 250-10x$ and when $dR/dx = 0$, $x = 25$ dollars.

16. Using the same method as in #15, $R = 100x +10[(25-x)/2]x$ or $R = 100x+125x-5x^2$. Then $dR/dx = 225-10x$. When $dR/dx = 0$, $x = \$22.50$.

CHAPTER 8, SECTION 6, SECOND SET

1. The cost function becomes $C(x) = 3x+2x$. The revenue is still xp where $p = 10-3x$. Hence the profit is $P(x) = xp-C = 10x-3x^2-5x = 5x-3x^2$. $P'(x) = 5-6x$ and this is a maximum when $x = 5/6$ and the price at which the commodity will be sold is $p = 10-3(5/6) = 15/2$.
2. Under no tax $P(x) = 10x-3x -3x^2= 7x-3x^2$. Then $P'(x) = 7-6x$ and $x = 7/6$. The price $p = 10-3(7/6) = 13/2$.
3. Now the consumer must pay 125% of what he previously paid. Hence the demand changes because the price is higher. The new demand function can be obtained from the old one by replacing p by (5/4)p so that $(5/4)p = 10-3x$ or $p = (4/5)(10-3x)$. (Of course since $3x = 10-(5/4)p$, for a given price p the demand will be less than under $3x = 10-p$.) The cost function for the producer remains the same. Hence $P(x) = x(4/5)(10-3x)-3x = 5x-(12/5)x^2$. Then $P'(x) = 5-(24/5)x$ and $P'(x) = 0$ for $x = 25/24$. The maximum profit is 125/48 and the corresponding price p is 11/2 without sales tax.
4. The solution follows that of exercise 3 except that 5/4 is replaced by 11/10. Thus the new demand function is $p = (10/11)(20-4x)$ and since $C(x) = 4x$, $P(x) = x(10/11)(20-4x)-4x$, $P'(x) = (200/11)-(80/11)x-4$. $P'(x) = 0$ when $x = 1.95$. The corresponding price p is \$11.10. The quantity sold comes from solving $p = (10/11)(20-4x)$ for x when $p = \$11.10$. x = approximately 2.

CHAPTER 8, SECTION 7

1. (a), (b) and (c). Find the relative maxima and minima and points of inflection and use these as aids to plotting.
   (d) $y' = \frac{5}{3}x^{2/3}$, $y'' = \frac{10}{9}x^{-1/3}$. No relative maxima or minima and no points of inflection. Point for point plotting with some attention to the behavior of y' and y'' is all one can apply.
   (e) $y' = -2x/(x^2+1)^2$. There is a relative maximum at $x = 0$. Since $y'' = (6x^2-2)/(x^2+1)^3$ there are points of inflection at $x = \pm\sqrt{1/3}$ and y approaches 0 as x approaches $+\infty$ and $-\infty$.
   (f) and (g) See (a), (b) and (c)
   (h) $y' = (-x^2+1)/(x^2+1)^2$. There is a relative minimum at $x = -1$ and a relative maximum at $x = +1$. $y'' = 2x(x^2-3)(x^2+1)/(x^2+1)^4$. There are points of inflection at $x = 0$, $x = \sqrt{3}$ and $x = -\sqrt{3}$
   (i) $y' = (9x-9)/(x+3)$. There is a relative minimum at $x = 1$. $y'' = 36/(x+3)^2$. There is no point of inflection. When x is very large and positive or negative, y is close to $+1$. As x approaches -3 from the left y approaches $+\infty$. As x approaches -3 from the right y approaches $+\infty$. Then y goes to a minimum of $-\frac{2}{16}$ at $x = 1$ and increases gradually to $y = 1$. The curve is always concave upward.
   (j) $y' = 4/(x+4)^2$. No relative maxima or minima. $y'' = -8/(x+4)^3$. There are no inflection points. y is infinite at $x = -4$. As x approaches -4 from the left y approaches $+\infty$. As x approaches -4 from the right y approaches $-\infty$. y approaches 2 as x approaches $+\infty$ or $-\infty$.
2. y'' is negative from A to C, positive from C to E, negative from E to G, positive from G to I.
3. Using the factored form of y we find $y'' = 2a(3x - r_1 - r_2 - r_3)$. Hence the inflection point is $x = (r_1 + r_2 + r_3)/3$.
4. The horizontal tangent occurs at $y' = 0$ or $3ax^2 + 2bx + c = 0$. The point of inflection occurs at $y'' = 0$ or $6ax + 2b = 0$. Since these equations hold at the same value of x, we take the value of x from the second one and substitute in the first one. Then $b^2 - 3ac = 0$.
5. $y'' = 12x^2-48x$. Hence $y'' = x(x-4)$. The zeros are clearly $x = 0$ and $x = 4$. Also y'' changes sign at $x = 0$ and $x = 4$.

Solutions to Chapter 9

CHAPTER 9, SECTION 2

1. Divide up the length from $x = 1$ to $x = 5$ into n equal parts Δx. Then we have $3\Delta x + 3(1 + \Delta x)\Delta x + 3(1 + 2\Delta x)\Delta x + \cdots + 3[1 + (n-1)\Delta x]\cdot\Delta x$ $= 3n\Delta x + 3(\Delta x)^2[1 + 2 + \cdots + (n-1)] = 3n\Delta x + 3(\Delta x)^2(n-1)n/2$. Since $\Delta x = 4/n$, we have $3n(4/n) + 3(4/n)^2(n^2/2 - n/2) = 12 + 24 - 24/n$. As n becomes infinite the limit is 36.
2. Divide up the length from $x = 1$ to $x = 5$ into n equal parts Δx. Then $\overline{S}_n$ $= 3(1 + \Delta x)\Delta x + 3(1 + 2\Delta x)\Delta x + \cdots + 3(1 + n\Delta x)\Delta x = 3n\Delta x$ $+ 3(\Delta x)^2[1 + 2 + \cdots + n] = 3n\Delta x + 3(\Delta x)^2 n(n+1)/2$. Since $\Delta x = 4/n$ we have $3n(4/n) + 3(4/n)^2(n^2/2 + n/2) = 12 + 24 + 24/n$. As n becomes infinite the limit is 36.
3. Divide up the interval from $x = 0$ to $x = 5$ into n equal parts. Then $\overline{S}_n$ $= (\Delta x)^2\Delta x + (2\Delta x)^2\Delta x + (3\Delta x)^2\Delta x + \cdots + (n\Delta x)^2\Delta x = (\Delta x)^3[1 + 4 + 9 + \cdots n^2]$ $= (\Delta x)^3(n^3/3 + n^2/2 + n/6)$. Since $x = 5/n$, we have $125(\frac{1}{3} + 1/2n + 1/6n^2)$. As n becomes infinite the limit is $^{125}/_{3}$.
4. Divide up the interval from $x = 1$ to $x = 5$ into n equal parts. Then using the smallest y-value in each subinterval, $\underline{S}_n = 1\Delta x + (1 + \Delta x)^2\Delta x$ $+ (1 + 2\Delta x)^2\Delta x + \cdots + [1 + (n-1)\Delta x]^2\Delta x$
$$= \Delta x[1 + 1 + 2\Delta x + (\Delta x)^2 + 1 + 4\Delta x + 4(\Delta x)^2 + \cdots + 1 + 2(n-1)\Delta x + (n-1)^2(\Delta x)^2]$$
$$= \Delta x[n + \Delta x(2 + 4 + \cdots + 2n - 2) + (\Delta x)^2(1 + 4 + \cdots + (n-1)^2)]$$
$= \Delta x[n + \Delta x(n-1)n + (\Delta x)^2(n^3/3 - n^2/2 + n/6)]$. Now $x = 4/n$. Hence we get $(4/n)[n + 4(n-1) + 16(n/3 - \frac{1}{2} + 1/6n) = 4 + 16 - 16/n + {}^{64}/_{3} - 64/2n + 64/6n^2]$. As n becomes infinite the limit is $^{124}/_{3}$.
5. Let us divide up the interval from $x = 0$ to $x = 5$ into 10 equal parts. Then $\overline{S}_{10} = (\Delta x)^2\Delta x + (2\Delta x)^2\Delta x + \cdots + (10\Delta x)^2\Delta x = (\Delta x)^3[1 + 4 + \cdots + 100]$. To get the sum of the squares from 1 to 100 we use (11) with $n = 10$. Then $s_{10} = {}^{1000}/_{3} + 50 + {}^{5}/_{3}$ and now, since $\Delta x = {}^{5}/_{10} = \frac{1}{2}$, $\overline{S}_{10} = (\frac{1}{2})^3({}^{1000}/_{3} + 50 + {}^{5}/_{3})$ $= {}^{385}/_{8}$.
6. Divide up the interval from $x = 0$ to $x = 1$ into 10 equal parts. Then
$$\underline{S}_n = 1\Delta x + \frac{1}{1 + (\Delta x)^2}\Delta x + \frac{1}{1 + (2\Delta x)^2}\Delta x + \cdots + \frac{1}{1 + (9\Delta x)^2}\Delta x.$$
Since $\Delta x = 0.1$, $\underline{S}_n = .1 + (1/1.01)(.1) + (1/1.04)(.1) + \cdots + (1/1.81)(.1)$. Then $\underline{S}_n = .1(1 + 1 + .\overline{96} + .9 + .86 + \cdots + .55) = 0.75$ approx.

CHAPTER 9, SECTION 3, FIRST SET

1. (a) $\frac{1}{3}, \frac{2}{3}, \ldots, \frac{5}{3}$; (b) 1, 4, 9, 16, 25; (c) $\frac{1}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}$;
   (d) $4, 5/\sqrt{2}, 6/\sqrt{3}, 7/\sqrt{4}, 8/\sqrt{5}$.
2. (c) $2 + (1/2^n)$; (d) $(-1)^{n+1}(1/n)$; (e) $1/n(n+1)$.
3. (b) It is easier to see the limit if one divides numerator and denominator by n. (c) Write as $(n+1)/n$ and divide numerator and denominator by n. (d) The magnitudes of the terms approach 1, but the signs alternate so that there is no limit. (e) The terms become infinite; no limit. (f) The terms approach 0. (g) Each term is 1; hence the limit is 1. (h) If we divide numerator and denominator by n we see that the fraction becomes infinite as n does. (i) Divide numerator and denominator by n. Then the fraction approaches 0. (j) 5.

CHAPTER 9, SECTION 3, SECOND SET

1. (a) The area bounded by the parabola $y = x^2$, the ordinates $x = 1$ and $x = 3$, and the x-axis.
   (b) The area bounded by $y = x^3$, the ordinate $x = 5$ and the x-axis.
   (c) The area bounded by the straight line $y = x+3$, the x-axis, and the ordinates at $x = 2$ and $x = 5$.
   (d) The area bounded by $y = x^2$ and the x-axis from $x = -1$ to $x = 4$.
   (e) The area bounded by the parabola $y = 9-x^2$ and the x-axis from $x = 1$ to $x = 3$.
   (f) The area bounded by the straight line $y = x-3$ and the x-axis from $x = 3$ to $x = 8$.
   (g) The area bounded by the upper half of the parabola $y^2 = x$, the x-axis and the ordinate at $x = 5$.
2. (d)

4. $\int_1^5 3x^2dx$. 5. $\int_{-2}^6 3x^2dx$. 6. $\int_2^{10} x\sqrt{x^2-2}\, dx$.

CHAPTER 9, SECTION 4

1. (a) $x^3/3\Big|_1^3 = 26/3$; (b) $x^4/4\Big|_0^1 = 1/4$; (c) $-x^{-1}\Big|_1^2 = 1/2$;
   (d) $x^3/3\Big|_{-3}^2 = 35/3$; (e) $3x^4/4\Big|_0^5 = 1875/4$; (f) $2\sqrt{5}\ x^{3/2}/3\Big|_0^{10} = 20\sqrt{50}/3$;
   (g) $x^3-x^2+5x\Big|_1^5 = 120$; (h) $(4x+1)^{3/2}/6\Big|_1^5 = (21^{3/2}-5^{3/2})/6$;
   (i) $5x^2/2-x^3/3\Big|_1^4 = 33/2$.
2. (a) $A = \int_2^6 x^2dx = x^3/3\Big|_2^6 = 69\ 1/3$; (b) $A = \int_4^8 x^2dx = x^3/3\Big|_4^8 =$
   $8^3/3-4^3/3 = 448/3$.

3. $A = \int_4^6 x\,dx = x^2/2\Big|_4^6 = 10$. The area of the trapezoid $= 1/2(2)(6+4) = 10$

4. $A = \int_3^6 9x\,dx = 9x^2/2\Big|_3^6 = 63/2$.

5. $A = \int_2^8 x^{1/3}dx = 3x^{4/3}/4\Big|_2^8 = 12 - 3\cdot 2^{1/3}/2$.

6. $A = \int_0^5 x^2dx = x^3/3\Big|_0^5 = 125/3$.

7. $A = \int_1^5 (x+1)^{1/2}dx = 2(x+1)^{3/2}/3\Big|_1^5 = 2(6^{3/2} - 2^{3/2})/3$.

8. $\lim\limits_{n\to\infty} S_n = \int_0^1 x^3\,dx$. By the fundamental theorem $A(x) = x^4/4$.
Then $A = x^4/4\Big|_0^1 = 1/4$

9. $\lim\limits_{n\to\infty} S_n = \int_1^5 3x^2\,dx$. Then $A(x) = x^3$ and $A = x^3\Big|_1^5 = 124$.

10. $\lim\limits_{n\to\infty} S_n = \int_{-2}^6 3x^2\,dx$. Then $A(x) = x^3$ and $A = x^3\Big|_{-2}^6 = 216 - (-8) = 224$.

11. $\lim\limits_{n\to\infty} S_n = \int_3^6 2x^2\,dx$. Then $A(x) = 2x^3/3$ and $A = 2x^3/3\Big|_3^6 = 126$.

12. $\lim\limits_{n\to\infty} S_n = \int_2^{10} \sqrt{x^2-2}\,x\,dx$. Then $A(x) = 1/3(x^2-2)^{3/2}$ and
$A = 1/3(x^2-2)^{3/2}\Big|_2^{10} = 1/3[(98)^{3/2} - 2^{3/2}] = 228\sqrt{2}$.

CHAPTER 9, SECTION 5

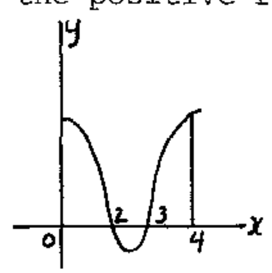
1. $\int_0^5 -3x\,dx = -\frac{3}{2}x^2\Big|_0^5 = -75/2$. The geometrical area is $75/2$.

2. (a) $\int_{-3}^4 3x\,dx = \frac{3}{2}x^2\Big|_{-3}^4 = \frac{3}{2}\cdot 16 - \frac{3}{2}\,9 = 21/2$.
(b) $\int_{-3}^0 3x\,dx + \int_0^4 3x\,dx = -\frac{27}{2} + 24$. However, the geometrical area $27/2 + 24$.

3. $\int_{-3}^3 -x^2dx = -x^3/3\Big|_{-3}^3 = -9-(+9) = -18$.
(b) Since the entire area lies below the x-axis the geometrical area is 18.

4. $\int_1^5 -(2x+1)^{1/2}dx = (-1/2)\int_1^5 (2x+1)^{1/2}2dx = (-1/3)(2x+1)^{3/2}\Big|_1^5 =$
$-(11^{3/2} - 3^{3/2})/3$. The geometrical area is the positive result.

5. $\int_0^4 (x-3)(x-2)(x+1)dx = \int_0^4 (x^3-4x^2+x+6)dx =$
$\int_0^2 - \int_2^3 + \int_3^4 = 22/3 - (-7/12) + 47/12 = 71/6$.



CHAPTER 9 , SECTION 6

1. $\int_1^4 (x^2 + x^3)\,dx = \int_1^4 x^2\,dx + \int_1^4 x^3\,dx = x^3/3 \Big|_1^4 + x^4/4 \Big|_1^4 = 84\tfrac{3}{4}$.

2. $\int_0^1 (9 - x^2)\,dx = \int_0^1 9\,dx - \int_0^1 x^2\,dx = 9x \Big|_0^1 - x^3/3 \Big|_0^1 = {}^{26}\!/_3$.

3. $\int_1^5 (x^3 + 9 - x^2)\,dx = \int_1^5 x^3\,dx + \int_1^5 9\,dx - \int_1^5 x^2\,dx = 156 + 36 - 41\tfrac{1}{3} = 150\tfrac{2}{3}$.

4. $BCDE = ACDF - ABEF = {}^5\!/_3\,(4^3 - 2^3)$
$- (4^3 - 2^3) = 37\tfrac{1}{3}$

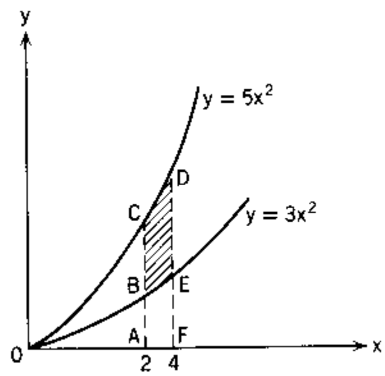


Fig. 1

5. $ABCD = -\tfrac{1}{3} + 1 = \tfrac{2}{3}$

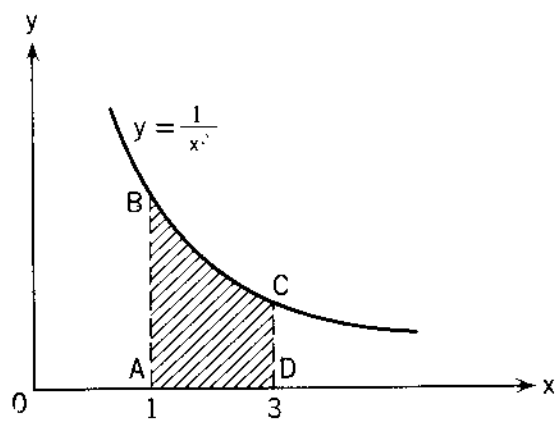


Fig. 2

6. $OABD = OABC - ODBC = (2^2 - 0^2) - \frac{1}{3}(2^3 - 0^3) = \frac{4}{3}$

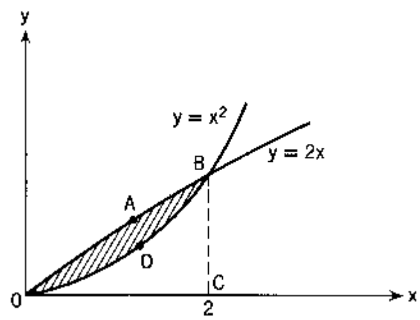


Fig. 3

7. $OABD = OABC - ODBC = \frac{2}{3}\sqrt{5}[(\sqrt[3]{5})^{3/2} - 0^{3/2}] - \frac{1}{3}[(\sqrt[3]{5})^3 - 0^3] = \frac{5}{3}$

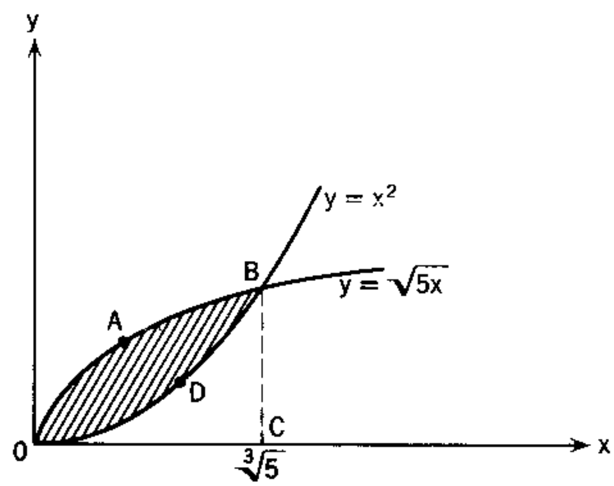


Fig. 4

8. $OBCD = ABCDE - ABODE = 18\sqrt{2}$

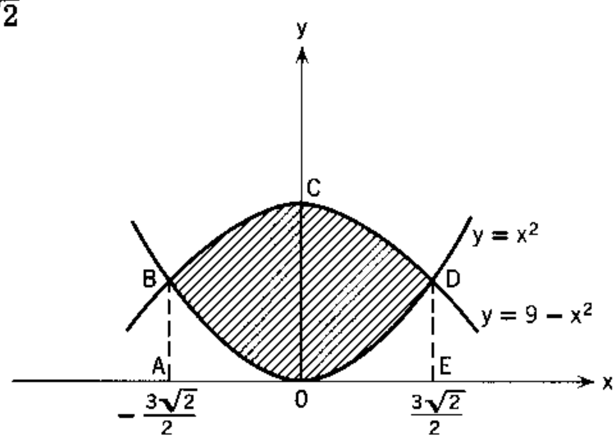
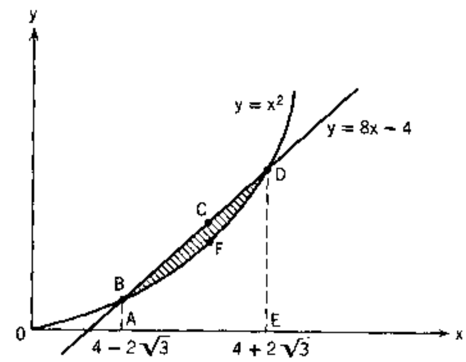


Fig. 5

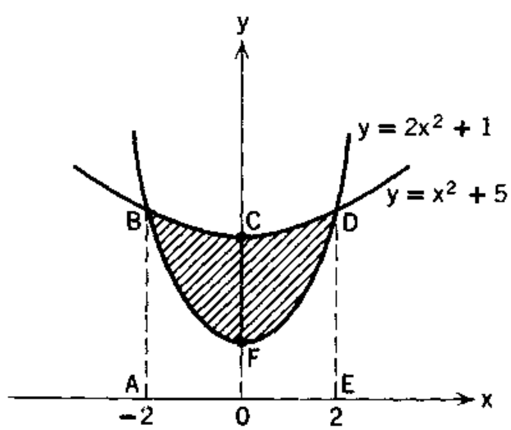
9. BCDF = ABCDE − ABFDE
$= 32\sqrt{3}$

Fig. 6



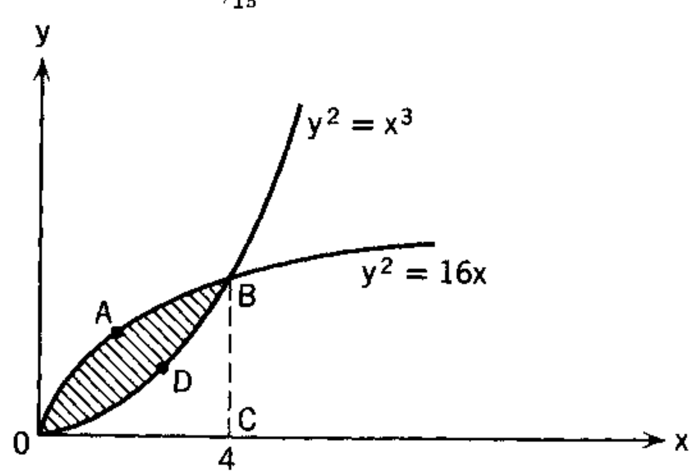
10. BCDF = ABCDE − ABFDE
$= 10^2/_3$

Fig. 7



11. OABD = OABC − ODBC = $8^8/_{15}$

Fig. 8

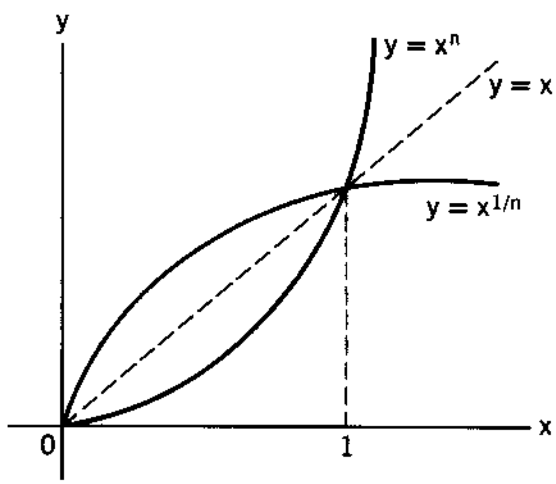


12. Since the curve of $y = x^3 - 8$ cuts the x-axis at $x = 2$, the area between $x = 1$ and $x = 2$ lies below the x-axis. This area is obtained from $A = x^4/4 - 8x$ by letting $x = 2$, then $x = 1$ and subtracting the second result from the first. This gives $-4\frac{1}{4}$. The area above the x-axis lies between $x = 2$ and $x = 3$. This area is $8\frac{1}{4}$. Hence the physical area is $12\frac{1}{2}$.

13. Let $F(x)$ be any indefinite integral of $y(x)$, where $y(x)$ is the given function of x. Then $\int_a^b cy\,dx = cF(x)\Big|_a^b = cF(b) - cF(a) = c[F(b) - F(a)] = c\int_a^b y\,dx$.

14. Let $u = x$ and $v = x^2$.

15. The function $y = x^n$ is shown in the figure. The graph of the inverse function $x = y^{1/n}$ is the graph of $y = x^n$. However when writing $y = x^{1/n}$ for the inverse function its graph is merely the interchange of x and



y in $x = y^{1/n}$. Hence the graph of $y = x^{1/n}$ is symmetric to that of $y = x^n$ about the line $y = x$. Then the area between $y = x^{1/n}$ and $y = x$ equals the area between $y = x$ and $y = x^n$. Hence the given area is twice the area of the triangle bounded by $y = x$, the x-axis and $x = 1$ or the area is 1.

16. We have to show that $\int_c^d x^a\,dx = (1/a)\int_{c^a}^{d^a} x\,dy$ where $x = y^{1/a}$. Here $x = y^{1/a}$ is the equation of the curve with y as the independent variable. We have but to integrate by the use of inverse of the power rule and the result follows.

CHAPTER 9, SECTION 7

1. 0. The definite integral is a constant.
2. By (29) the answer is x^3. If we evaluate we get $u^4/4\Big|_a^x = x^4/4-a^4/4$ and by differentiating with respect to x we again get x^3.
3. By (29) $dg/dx = \sqrt{x^2+2}$. Then $d^2g/dx^2 = (1/2)(x^2+2)^{-1/2}(2x)$.
4. Let $z = x$. Then $g(z) = \int_0^z f(u)du$. $dg/dx = (dg/dz)(dz/dx)$ $= f(z)2x = f(x^2)2x$. Check by letting $f(u) = x$ to convince students.
5. One cannot regard an area as a sum of line segments.

CHAPTER 9, SECTION 8, FIRST SET

1. Here $h = 1$ and $n = 4$. Also $y_0 = f(1) = 1$, $y_1 = f(2) = 4$, y $y_2 = f(3) = 9$, $y_3 = f(4) = 16$ and $y_4 = f(5) = 25$. Substitution in (32) yields 42. Use of the fundamental theorem gives
$$\int_1^5 x^2dx = x^3/3\Big|_1^5 = 41^1/_3.$$
2. Here $h = .1$. Then $y_0 = f(0) = 1$, $y_1 = f(.1) = 1/1.01$, $f(.2) = 1/1.04$, $f(.3) = 1/1.09$, $f(.4) = 1/1.16$, $f(.5) = 1/1.25$. Substitution in (32) yields 0.463.
3. Here $h = 1/2$ and $y_0 = f(0) = 1$, $y_1 = f(1/2) = 1.06$, $f(1) = 1.41$, $f(3/2) = 2.09$, $f(2) = 3$, $f(5/2) = 4.08$, $f(3) = 5.29$. Substitution in (32) yields 7.39.
4. Here the h is evidently 1 and there are 10 subintervals. The values of $f(0), f(1),\cdots,f(10)$ are given. Hence we can substitute at once in (32). The result is 10.06.

CHAPTER 9, SECTION 8, SECOND SET

1. Here $n = 4$ and $h = .25$. $y_0 = f(0) = 1$, $y_1 = f(.25) = 1/1.25$, $y_2 = f(.5) = 1/1.5$, $y_3 = f(.75) = 1/1.75$, $y_4 = f(1) = 1/2$. Substitution in (36) yields $(.25/3)[1+1/2+2(1/1.5)+4(1/1.25+1/1.75)] = .693$.
2. Here $n = 4$ and $h = 1$. $y_0 = f(1) = 1$, $y_1 = f(2) = 4$, $y_2 = f(3) = 9$, $y_3 = f(4) = 16$ and $y_4 = f(5) = 25$. Substitution in (36) gives $(1/3)[1+25+2(9)+4(4+16)] = 124/3$. The exact value is also 124/3 because we are dealing with a parabola $y = x^2$ to start with and Simpson's rule fits a parabola to each arc of $y = x^2$.
3. We use (36). In this example $h = 1/2$ and $n = 6$. The y-values are $y_0 = f(0) = 1$, $y_1 = f(.5) = 1.06$, $y_2 = f(1) = 1.41$, $y_3 = f(1.5) = 2.09$, $y_4 = f(2) = 3$, $y_5 = f(2.5) = 4.08$, $y_6 = f(3) = 5.29$. Substitution in (36) yields $1/6(44.03) = 7.34$.
4. Here $n = 6$ and $h = 1$. The function values are given by the table. Thus $y_0 = 32$, $y_1 = 38,\cdots,y_6 = 38$. Substitution in (36) yields 37.33.

5. Here $n = 4$ and $h = 1/8$. Here $y_0 = f(0) = 1$, $y_1 = f(1/8) = 1/(1+1/64)$, $y_2 = f(1/4) = 1/(1+1/16)$, $y_3 = f(3/8) = 1/(1+9/64)$, $y_4 = f(1/2) = 1/(1+1/4)$. Using (36) gives $(1/24)[1+4/5+2(16/17)+4(64/65+64/73)] = 0.464$.

Solutions to Chapter 10

CHAPTER 10, SECTION 2

1. (a) A sine curve with period $2\pi/3$ and amplitude of 1.
   (b) A sine curve with period 2π and amplitude of 3.
   (c) A sine curve with period π and amplitude of 3
   (d) A sine curve with period $\pi/2$ and amplitude of 2
   (e) A sine curve which is displaced $\pi/2$ units to the left of the normal $y = \sin x$.
   (f) A sine curve which is raised $\pi/2$ units above the normal sine curve
   (g) A sine curve which is displaced one unit to the right of $y = \sin x$.
   (h) Graph $y = 2 \sin 3x$ and turn it $180°$ about the x-axis.
3. Sketch $y = x$ and $y = -x$. The final curve oscillates between these two lines with zeros at the usual zeros of $y = \sin x$.
5. Sketch $y = x^2$ and $y = \sin x$ and then add ordinates at a number of values of x.

CHAPTER 10, SECTION 3

1. By following the method indicated we have $(1 - \cos^2 x)/x(1 + \cos x)$ $= \sin^2 x/x(1 + \cos x)$. Then $\lim\limits_{x\to 0} \sin^2 x/x(1 + \cos x) = \lim\limits_{x\to 0} \sin x/x \cdot$ $\lim\limits_{x\to 0} \sin x/(1 + \cos x) = 1 \cdot 0 = 0$.
2. (a) $\lim\limits_{x\to 0} \sin 2x/x = \lim\limits_{x\to 0} 2 \sin 2x/2x = 2 \lim\limits_{x\to 0} \sin z/z$ with $z = 2x$. Since $z \to 0$ as $x \to 0$, the answer is 2.
   (b) Use the same method as in (a) except that a replaces 2.
   (c) Think of Δx as just a variable z. Then the limit is 1.
   (d) $\lim\limits_{x\to 0} (1 - \cos x)/x^2 = \lim\limits_{x\to 0} [(1 - \cos x)(1 + \cos x)]/x^2(1 + \cos x)$
   $\lim\limits_{x\to 0} \sin^2 x/x^2(1 + \cos x) = \lim\limits_{x\to 0} \sin x/x \cdot \lim\limits_{x\to 0} \sin x/x \quad \lim\limits_{x\to 0} 1/(1 + \cos x)$
   $= 1 \cdot 1 \cdot \frac{1}{2}$.
   (e) $\lim\limits_{x\to 0} \tan x/x = \lim\limits_{x\to 0} \sin x/x \cos x = \lim\limits_{x\to 0} \sin x/x \quad \lim\limits_{x\to 0} 1/\cos x = 1 \cdot 1$.

CHAPTER 10, SECTION 4

1. (a) Let $u = 2x$ and apply the chain rule.
   (b) Let $u = 5x$ and apply the chain rule. Ans. $y' = -5 \sin 5x$.
   (c) The constant 3 merely multiplies the derivative of $y = \cos 2x$. Let $u = 2x$ and apply the chain rule.

(d) Let $u = 5x$. Then $y = 5 \tan u$. Apply the chain rule. Ans. $y' = 30 \sec^2 5x$.
(e) Let $u = 4x$ and apply the chain rule.
(f) Differentiate as a product of two functions. Ans. $y' = -\sin^2 x + \cos^2 x$.
(g) Let $u = 2x$. Then $y = 1/\cos 2u$. Differentiate with respect to u and multiply by du/dx.
(h) Since $\cot x = \cos x/\sin x$, $y = \cos x$ and $y' = -\sin x$.
(i) Since $\tan x = \sin x/\cos x$, $y = \sin x$ and $y' = \cos x$.
(j) $1 - \cos^2 x = \sin^2 x$. Hence $y = \sin x$ and $y' = \cos x$.
(k) $1 + \tan^2 x = \sec^2 x$.
(l) Let $u = 2 - \cos^2 x$ and apply the chain rule. Then $y' = \frac{1}{2} u^{-1/2} \cdot d(2 - \cos^2 x)/dx$. Now let $v = \cos x$. Then $y' = \frac{1}{2} u^{-1/2} \cdot (2 \cos x \sin x)$. The quantity $2 - \cos^2 x = 1 + 1 - \cos^2 x = 1 + \sin^2 x$.

2. (a) Let $u = \sin x$. Then $y = u^3$.
(b) Differentiate as a product and use (a) to handle $\sin^3 x$. Ans. $y = \sin^2 x(3 \cos^2 x - \sin^2 x)$.
(c) Let $u = \sin 2x$. Then $y = u^3$. To find du/dx apply the chain rule again to $\sin 2x$.
(d) Write $y = \sin^{1/3} x$ and let $u = \sin x$. Ans. $y' = \cos x/3(\sin x)^{2/3}$.
(e) As in (c), let $u = \cos 2x$. Then $y = u^2$. To differentiate u apply the chain rule again to $\cos 2x$.
(f) Let $u = x^3$. Then $y = \sin u$ and $y' = \cos u(3x^2) = 3x^2 \cos x^3$.
(g) Since $\cot 2x = 1/\tan 2x$, $y = 1$.
(h) Let $u = 1/x$. Then $y = \sin u$. $y = \cos u\ (-1/x^2) = (-1/x^2) \cos (1/x)$.
(i) Treat as a product. To differentiate $\cos(1/x)$ use the method of (h).
(j) Let $u = \sin x^3$. Then $y = u^{1/2}$. To find du/dx apply (f). Ans. $y' = 3x^2 \cos x^3/2\sqrt{\sin x^3}$.
(k) Let $u = \sin x$. Then $y = \cos u$.
(l) $y = 1$. Hence $y' = 0$.
(m) Since $\sin 2x = 2 \sin x \cos x$, $y = 2 \cos^2 x$. Let $u = \cos x$.

3. If we use the identity we have $\Delta y/\Delta x = [2 \cos (x_0 + \Delta x/2) \sin (\Delta x/2)]/\Delta x = \cos (x_0 + \Delta x/2)[\sin (\Delta x/2)]/(\Delta x/2)$. Then $y' = \cos x_0$.
4. Let $u = \pi/2 - x$. Then $y = \sin u$. Apply the chain rule. Then $y' = \cos u(-1)$. But $\cos(\pi/2 - x) = \sin x$. Then $y' = -\cos x$.
5. Differentiate the quotient by the theorem on a quotient of two functions. The result is (20).
6. Same method as in Exercise 3. The result is (23).
7. Same method as in Exercise 3. The result is (26).
8. Same method as in Exercise 3. The result is (29) or the last line on p. 240.
10. $y' = -\sin x$; $y'' = -\cos x$.
11. $R = (V^2/16)(\cos^2 A - \sin^2 A)$. $R' = 0$ implies $\cos^2 A = \sin^2 A$ and so $A = \pi/4$.
12. If we form $\Delta y/\Delta x$ for the function $y = \sin 2x$ we get the expression in the text. The limit as Δx approaches 0 is the definition of the derivative of $\sin 2x$. Hence $y' = 2 \cos 2x$.

13. If we express $\Delta y/\Delta x$ for the function $y = \sin x$ at the value $x_0 = \pi/2$ of x we obtain $\{\sin[(\pi/2) + \Delta x] - \sin(\pi/2)\}/\Delta x$. Now let $\Delta x = x - \pi/2$. Hence we get the expression in the text. However limit of $\Delta y/\Delta x$ as Δx approaches is the derivative of $y = \sin x$ at $x = \pi/2$. Hence the answer is $\cos(\pi/2)$ or 0.

14. The mass m is pulled downward with a force (its weight) of 32m. This force is transmitted directly to the mass M and pulls it upward. However (see (33) of Chap. 3) there is an acceleration of $32 \sin A$ and therefore, by Newton's second law, a force of $32M \sin A$ pulling the mass M down the plane. Hence the net upward force on M is $32m - 32M \sin A$. The net upward acceleration by $F = (M+m)a$, $a = (32m-32M \sin A)/(M+m)$. If we let x represent the variable distance up the plane measured from the bottom then, since $a = d^2x/dt^2$, we integrate and apply the initial condition $\dot{x} = 0$ when $t = 0$. Integrate again and apply the initial condition $x = 0$ when $t = 0$. This gives $x = 16(m-M \sin A)t^2/(M+m)$. The length of the plane to be covered is $h/\sin A$. Then $h/\sin A = 16(m-M \sin A)t^2/(M+m)$. Thus the time to travel up the plane is $t = \sqrt{h(M+m)}/4[\sin A(m-M \sin A]^{-1/2}$. If we now find dt/dA (by letting u = the quantity in the brackets and applying the chain rule we find from $dt/dA = 0$ that $\sin A = m/2M$. There is another possible root, namely $\cos A = 0$ and $A = 90°$ but this answer cannot be considered because it gives an imaginary value for t unless $m > M$. But m need not be $> M$; it need only be greater than $M \sin A$ to provide an acceleration up the plane. Moreover $A = 90°$ is another situation entirely.

The force F acting on the whole system acts on both masses m and M and this is why we must write $F = (m+M)a$. Another way to see this is to take into account the tension T in the string. The forces acting on M are $T-32M \sin A$, and hence Newton's second law says (1) $T-32M \sin A = Ma$. The total forces acting on m are $32m-T$, and by Newton's second law (2) $32m-M = ma$. Hence by adding (1) and (2) (3) $32m-32M \sin A = (M+m)a$. From (3) we see that the acceleration of the whole system is $a = (32m-32M \sin A)/(M+m)$.

15. Let A be the angle of inclination of the desired straight line (Fig. 1). A is measured from the horizontal distance d clockwise and so is also the inclination of the inclined line on which the particle slides. Then if x is measured from the point along the inclined line, $\ddot{x} = 32 \sin A$ and since $\dot{x} = 0$ when $t = 0$ and $x = 0$ when $t = 0$, $x = 16t^2 \sin A$. Then $t = \frac{1}{4}\sqrt{x/\sin A}$. But the distance x to be covered is $d/\cos A$. Hence $t = \frac{1}{4}\sqrt{d/\sin A \cos A} = (\sqrt{d}/4)(\sin A \cos A)^{-1/2}$. Now find dt/dA (by letting $u = \sin A \cos A$). When $dt/dA = 0$, $\cos^2 A = \sin^2 A$ and $A = \pi/4$.

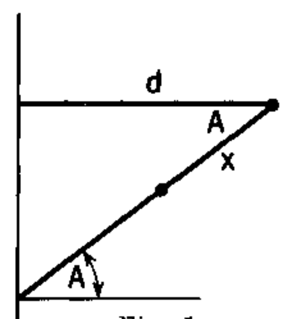


Fig. 1

16. Picture the pendulum making an angle A with the vertical. Then the height of the pendulum above the level of its lowest position is $h = 4 - 4 \cos A$. Hence $dh/dt = 4 \sin A \, dA/dt$. At $A = 30°$ and for $dA/dt = 18° = \pi/10$ rad. per sec., $dh/dt = \pi/5$ ft/sec.

17. If A is the angle of elevation of the plane at any time and x is the horizontal distance traveled by the plane (measured from directly above the observer) when the elevation is A, then $x = 2 \cot A$. Hence $dx/dt = -2 \csc^2 A \, dA/dt$. When $A = 30°$ and $dA/dt = -15° = -\pi/12$ rad/sec, $dx/dt = 2\pi/3$ mi/min.
18. Measuring the distance, x, along the shore from the foot of the perpendicular from the beacon to the shore and taking A as the angle between this perpendicular and the beam, we have $x = 3600 \tan A$. Hence $dx/dt = \dot{x} = 3600 \sec^2 A \cdot \dot{A}$. Now $\dot{A}$ is constant and is 4π rad/min. Hence $\dot{x} = 14{,}400\ \pi/\cos^2 A$.
    (a) Here $A = 0$ and $\cos A = 1$.
    (b) Here $\cos A = {}^{3600}/_{4800} = {}^{3}/_{4}$.
19. Take A to be the angle which the line from the center of the wheel to the cab makes with the vertical from the center of the wheel to the ground. Then the height of the passenger above the ground is $h = 30 - 25 \cos A$. Hence $\dot{h} = 25(\sin A)\dot{A}$. When $h = 40$, $\cos A = -{}^{10}/_{25} = -{}^{2}/_{5}$, and $\sin A = \sqrt{21}/5$. (When h is increasing and above 30, A is a second quadrant angle.) $\dot{A}$ is π rad/min. Then $\dot{h} = 25(\sqrt{21}/5)\pi$.
20. We use the suggestion that we need consider only those situations in which the destroyer heads straight for the battleship. However, the destroyer does not know at what angle ϕ to head. Let us determine ϕ by the condition that $C'D$ is to be a minimum. $C'D = AD - AC' = \sqrt{4 + 100t^2} - 8t$. If we differentiate with respect to t and set the derivative equal to 0 we get $t = {}^{4}/_{15}$. To find what ϕ is, we have that at this value of t, $BD = 10t = {}^{8}/_{3}$. Then $\tan\phi = BD/AB = {}^{4}/_{3}$ and $\sin\phi = 0.8$.
21. From Fig.10-12, $x = OA \cos\theta = (27 - OB)\cos\theta$. But $OB = 8/\sin\theta$. Then $x = [27 - (8/\sin\theta)]\cos\theta = 27\cos\theta - 8\cot\theta$. We find $dx/d\theta$ and set it equal to 0. This gives $\sin\theta = {}^{2}/_{3}$. Then $\cos\theta = \sqrt{5}/3$ and the minimum possible value of x is $5\sqrt{5}$ ft.

CHAPTER 10, SECTION 5

1. (a) Write $y' = {}^{1}/_{3} \cos(3x)3$ and let $u = 3x$. Use (38).
   (b) Write $y' = {}^{1}/_{4} \sin(4x)4$ and let $u = 4x$. Answer. $y = -{}^{1}/_{4} \cos 4x + C$.
   (c) Use $\sin x = \sqrt{(1 - \cos 2x)/2}$. Then $y' = {}^{1}/_{2} - {}^{1}/_{2} \cos 2x = {}^{1}/_{2} - {}^{1}/_{4} \cos(2x)2$. In the second term let $u = 2x$ and use (38). Integrate and use $\sin 2x = 2 \sin x \cos x$ to get the text's answer.
   (d) The method is the same as in (c) except $\cos = \sqrt{(1 + \cos 2x)/2}$. Ans. $y = x/2 + (\sin x \cos x)/2 + C$.
   (e) Let $u = \sin x$. Then $y' = u^2\, du/dx$.
   (f) Let $u = \sin 3x$. Then $y' = u^2 \cos 3x$. We need $3 \cos 3x$ for our du/dx. Hence write $y' = {}^{1}/_{3} u^2 \cos 3x \cdot 3 = {}^{1}/_{3} u^2\, du/dx$. Then $y = {}^{1}/_{9} \sin^3 3x + C$.
   (g) Write $y' = \sec^2 x \sec x \tan x$ and let $u = \sec x$. Then we have the proper du/dx to apply the inverse of the power rule.

(h) Let $u = \cot x$. Then $du/dx = -\csc^2 x$. Then $y' = -u^4 du/dx$. Hence $y = -(\cot^5 x/5) + C$.

(i) Let $u = \tan 2x$. Then $du/dx = 2 \sec^2 2x$. Write $y' = \frac{1}{2} u^4 du/dx$. Then y is as in the text.

(j) Write $y' = \cot^{-3} 2x \csc^2 2x$. Let $u = \cot 2x$. Then $du/dx = -2 \csc^2 2x$. Hence write $y' = -\frac{1}{2} u^{-3} du/dx$. Then $y = \frac{1}{4} \cot^{-2} 2x + C = \frac{1}{4} \tan^2 2x + C$.

(k) Write $y' = \cot^{3/2} x \csc^2 x$. Let $u = \cot x$. Then $du/dx = -\csc^2 x$. Hence write $y' = -u^{3/2} du/dx$ and integrate.

(l) Write $y' = \csc^3 ax \csc ax \cot ax$. Let $u = \csc ax$. Then $du/dx = -a \csc ax \cot ax$. Hence write $y' = -(1/a) u^3 du/dx$. Then $f(x) = -1/(4a) \csc^4 ax + C$.

(m) Write $y' = \sec^4 x \sec x \tan x$ and let $u = \sec x$.

(n) Write $y' = (\tan x + 3)^{-1/2} \sec x$. Let $u = \tan x + 3$. Then $du/dx = \sec^2 x$. Hence $y = 2(\tan x + 3)^{1/2} + C$.

(o) Let $u = x^2$. Then $f'(x) = (\cos u)x = 1/2(\cos u)2x = 1/2 \cos u \, du/dx$. Now use (38). Hence $f(x) = 1/2 \sin u + C = 1/2 \sin x^2 + C$.

2. The constant of integration has been ignored. $y = (\sin^2 x)/2 + C$ in one case and $y = -(\cos^2 x)/2 + C'$ in the other. The two constants differ in value. In fact if we let $C' = C + \frac{1}{2}$ we get the first answer.

3. Yes. Whatever behavior $f(x)$ has in one period it will have in another. In particular $f'(x)$ will be the same in the two periods because $f'(x)$ depends only on the values of $f(x)$.

4. $y' = 1 + \cos x$ is periodic. Its graph is that of $\cos x$ but raised one unit above the x-axis. However $y = x + \sin x$ and this is not periodic. Exercises 1(c) and 1(d) above give other examples.

5. $A = \int_0^\pi \sin x dx = -\cos x \Big|_0^\pi = -(-1)+1 = 2.$

6. $A = \int_0^{\pi/2} \cos x \, dx - \int_{\pi/2}^{3\pi/2} \cos x \, dx + \int_{3\pi/2}^{2\pi} \cos x \, dx =$

$\sin x \Big|_0^{\pi/2} - \sin x \Big|_{\pi/2}^{3\pi/2} + \sin x \Big|_{3\pi/2}^{2\pi} = 1-0-(-1-1)+(0-(-1)) = 4$

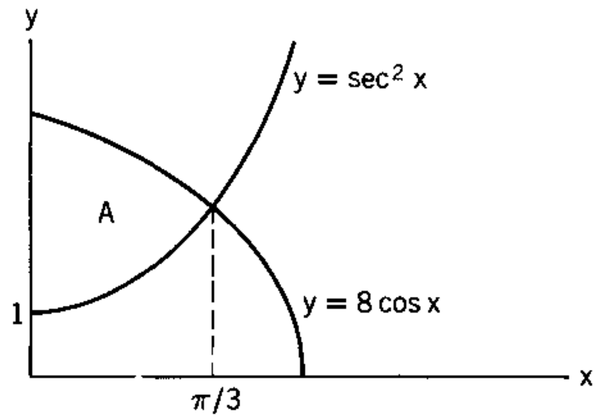
7. The two curves $y = \cos x+1$ and $y = 3/2$ intersect at $\cos x = 1/2$ so that $x = \pi/3$, $y = 3/2$ are the coordinates of the point of intersection. Then the area must be broken up into two points thus:

$A = \int_0^{\pi/3} [(\cos x+1)-3/2]dx + \int_{\pi/3}^{\pi} [3/2-(\cos x+1)dx = \sin x - x/2 \Big|_0^{\pi/3}$

$+ x/2 - \sin x) \Big|_{\pi/3}^{\pi} = \sqrt{3}/2 - \pi/6 + (\pi/2 - \pi/6 + \sqrt{3}/2) = \sqrt{3} + \pi/6.$

8. $\int_0^{\pi/2} \sin^2 x \, dx = \int_0^{\pi/2} (1-\cos 2x)dx/2 = x/2 - 1/4 \sin 2x \Big|_0^{\pi/2} = \pi/4.$

9. We must find the point of intersection of $y = \sec^2 x$ and $y = 8 \cos x$. Solving simultaneously gives $x = \pi/3$. The area under $y = 8 \cos x$ from $x = 0$ to $x = \pi/3$ is given by integrating $A' = 8 \cos x$. Then $A_1 = 8 \sin x$. Now substitute $\pi/3$ and 0 and subtract the second result from the first. Then $A_1 = 4\sqrt{3}$. The area under $y = \sec^2 x$ from $x = 0$ to $x = \pi/3$ is obtained from $A' = \sec^2 x$. Then $A_2 = \tan x$. Again substitute $\pi/3$ and 0 and subtract the second from the first. Since $\tan(\pi/3) = \sqrt{3}$, $A_2 = \sqrt{3}$. Then $A = A_1 - A_2 = 3\sqrt{3}$.



10. The error is replacing $\sqrt{(1-\cos x)/2}$ by $+\sin(x/2)$ in the entire interval $(0,4\pi)$. In $(2\pi,4\pi)$, $\sin(x/2)$ is negative and one must use $-\sin(x/2)$.

CHAPTER 10, SECTION 6, FIRST SET

1. At $t = 0$ we must have $-D = C \sin\phi$. Since $\dot{y} = C\sqrt{k/m} \cos(\sqrt{k/m}\,t + \phi)$ we must have at $t = 0$, $0 = C\sqrt{k/m} \cos\phi$. The right side can be 0 if C is 0 or $\phi = \pi/2$. But $C = 0$ will not satisfy the first condition. Hence $\phi = \pi/2$ and $C = -D$ satisfies both initial conditions.
2. At $t = 0$, y must be 0. From (55) we have $0 = B \cos 0$. Hence $B = 0$. Then so far $y = A \sin\sqrt{k/m}\,t$. Then $\dot{y} = A\sqrt{k/m} \cos\sqrt{k/m}\,t$. At $t = 0$, $\dot{y} = 5$. Then $A = 5\sqrt{m/k}$. With these values of A and B, (55) reduces to $y = 5\sqrt{m/k} \sin\sqrt{k/m}\,t$ and the amplitude is $5\sqrt{m/k}$.
3. To write (61) in the form (55) we rewrite (61) as $y = \sqrt{v_0^2 m/k + D^2}[(v_0/\sqrt{v_0^2 m/k + D^2}) \sin\sqrt{k/m}\,t - (D/\sqrt{v_0^2 k/m + D^2}) \cos\sqrt{k/m}\,t]$. Now we introduce an angle ϕ whose cosine is $v_0\sqrt{v_0^2 m/k + D^2}$ and whose sine is $D/\sqrt{v_0^2 m/k + D^2}$. Then $y = \sqrt{v_0^2 m/k + D^2} \sin(\sqrt{k/m}\,t - \phi)$. The amplitude is the radical.
4. From (60) we have $\dot{y} = D\sqrt{k/m} \sin\sqrt{k/m}\,t$. This velocity $\dot{y}$ is greatest when the sine function is 1 or $\sqrt{k/m}\,t = \pi/2 + 2n\pi$, where n is any integer. Then $t = \sqrt{m/k}\,(\pi/2) + 2n\pi\sqrt{m/k}$. The acceleration, given by $\ddot{y}$, is greatest when the cosine function is 1 or $\sqrt{k/m}\,t = 0 + 2n\pi$, or when $t = 2n\pi\sqrt{m/k}$.

5. If gravity is ignored the spring is not extended by the addition of the mass m to the lower end. Then (50) is irrelevant and (51) remains the same except that y now means displacement from the end of the unextended spring. Since (51) remains the same the solution is still (52).
6. We may start from (52) and then use the initial conditions. The only difference from what was done on p. 253 in applying the initial conditions there is that in the present exercise $y = 2$ in. or $\frac{1}{6}$ ft when $t = 0$. Hence in place of (60) we get $y = \frac{1}{6}\cos\sqrt{k/m}\,t$. We are also told that $m = \frac{1}{4}$ and $d = \frac{1}{3}$ ft. Hence by (50), $k(\frac{1}{3}) = 32(\frac{1}{4})$ or $k = 24$. Hence the final formula is $y = \frac{1}{6}\cos\sqrt{96}\,t$.
7. We may first of all determine k by the use of (50). When $m = 1$, $d = \frac{1}{24}$ ft. Hence by (50), $k = 768$. When the mass of 3 pounds is attached to the spring, it will stretch $1\frac{1}{2}$ in. or $\frac{3}{24}$ ft. Hence the equilibrium position is $\frac{3}{24}$ ft. below the lower end of the unextended spring. By placing the mass on the spring and then releasing it suddenly we are fixing the initial condition that $y = \frac{3}{24}$ when $t = 0$. Hence by arguing as in Exercise 6 or by using (60) with $-D$ replaced by $\frac{3}{24}$, with $k = 768$ and $m = 3$ we have $y = \frac{3}{24}\cos\sqrt{256}\ t = \frac{1}{8}\cos 16t$. The period of the motion is $2\pi/16$ or about 0.393 sec. The mass will fall until it reaches the lowest point on the cosine curve. Since the amplitude is $\frac{1}{8}$ it will fall $\frac{2}{8}$ ft or 3″.
8. As in Exercise 7, $k = 768$. The initial conditions here are $y = 0$ when $t = 0$ and $\dot{y} = 1$ when $t = 0$. If we use (52) and apply the condition $y = 0$ when $t = 0$ we get $B = 0$. Then, so far, $y = A\sin\sqrt{768/4}\,t$. Then $\dot{y} = A\sqrt{192}\cos\sqrt{192}\,t$. When $t = 0$, $\dot{y} = 1$. Hence $A = 1/\sqrt{192}$. Then $y = (1/\sqrt{192})\cos\sqrt{192}\,t$.
9. Suppose, as Fig 10-16 shows, the particle is a distance x, to the right of 0. Then the pull to the left is the amount of stretch in the left hand portion of the string. When the particle is at 0 the stretch (extension over the normal length ℓ) is already $a - \ell$ and by pulling the particle a distance x more to the right the stretch is $a - \ell + x$. If the "spring" constant is k, then by Hooke's law the force pulling the particle to the left is $k(a - \ell + x)$. Similarly, the stretch of the right hand portion is $a - \ell - x$ and the force pulling the object to the right is $k(a - \ell - x)$. The net force (in the positive x-direction) is $k(a - \ell - x) - k(a - \ell + x) = -2kx$. By Newton's second law, $m\ddot{x} = -2kx$. If we compare this equation with (51) we see that it is of the same form with $2k$ replacing k. Hence we may use (52) with $2k$ replacing k and $y = A\sin\sqrt{2k/m}\,t + B\cos\sqrt{2k/m}\,t$. Now if the particle is pulled initially a distance D to the right, say, and then released, we have $y = D$ when $t = 0$ and $\dot{y} = 0$ when $t = 0$. Then as on p. 253 we can determine that $A = D$ and $B = 0$.
10. If we let θ be the angular displacement then θ is of the form $\theta = a\sin bt$. We are told that $a = \pi/2$ and $2\pi/b = \frac{1}{2}$ so that $b = 4\pi$. Then $\theta = (\pi/2)\sin 4\pi t$. Then $\dot{\theta} = 2\pi^2\cos 4\pi t$ and $\ddot{\theta} = -8\pi^3\sin 4\pi t$. When $t = 2$, $\dot{\theta} = 2\pi^2$ rad/sec and $\ddot{\theta} = 0$.

CHAPTER 10, SECTION 6, SECOND SET

1. This equation is of the form (51) or (65). If we use (51) we see that k in the present case replaces k/m there.
2. We use (66). We have as our initial conditions that $\theta = 0.1$ when $t = 0$ and $\dot{\theta} = -0.05$ when $t = 0$. The first condition substituted in (66) gives $B = 0.1$. To meet the second condition we first obtain from (66) that $\dot{\theta} = \sqrt{32/\ell}\, A\cos\sqrt{32/\ell}\,t - \sqrt{32/\ell}\, B\sin\sqrt{32/\ell}\,t$. Since $\dot{\theta} = -0.05$ when $t = 0$ $A = -0.05\sqrt{\ell/32}$. Then $\theta = -0.05\sqrt{\ell/32}\cdot\sin\sqrt{32/\ell}\,t + 0.1\cos\sqrt{32/\ell}\,t$.
3. We use (66) with the initial conditions $\theta = 0$ when $t = 0$ and $\dot{\theta} = 0.1$ when $t = 0$. Now follow the method of Exercise 2. We find that $B = 0$ and $A = 0.1\sqrt{\ell/32}$.
4. Since $s = \ell\theta$ (p.293) we have, by (66), $s = \ell\, A\sin\sqrt{32/\ell}\,t + \ell\, B\cos\sqrt{32/\ell}\,t$.
5. From (70) we have $\dot{\theta} = -0.1\sqrt{32/\ell}\,\sin\sqrt{32/\ell}\,t$. Hence the angular velocity varies sinusoidally with an amplitude of $-0.1\sqrt{32/\ell}$ and the period is $2\pi\sqrt{\ell/32}$.
6. The period is given by (67). To make T twice as large we must increase ℓ by a factor of 4.
7. The period does not depend on the mass of the bob.
8. When the bob is at its highest point, its linear and angular velocities are 0. Hence the bob is momentarily stationary. Moreover its velocity is small near that point of its path.
9. The amplitude is $\pi/16$ and the period is $8/3$ seconds. From (67) we see that $\sqrt{\ell/32} = 4/3\pi$ so that $\sqrt{32/\ell} = 3\pi/4$. If we now assume that the pendulum started from the equilibrium position with some initial velocity then we may use the form $\theta = A\sin\sqrt{32/\ell}\,t$. (Compare Exercise 3.) Then
   (a) $\theta = (\pi/16)\sin(3\pi t/4)$
   (b) The angular velocity is $\dot{\theta} = (3\pi^2/64)\cos(3\pi t/4)$. The maximum value occurs at $t = 0$ or any multiple of 2π. At such values $\dot{\theta}$ is $3\pi^2/64$.
10. As pointed out in the text, $\dot{s} = \ell\dot{\theta}$. Then from (70) $\dot{s} = -\ell(0.1)\sin\sqrt{32/\ell}\,t$.

CHAPTER 10, SECTION 6, THIRD SET

1. We saw in the derivation in the text that the component of the force of gravity which causes the motion (p.298) is $PQ = kmx$. In a longer tunnel x varies over a longer range of values and the motion starts out with a larger acceleration. Hence the object acquires more velocity. The path is longer but the greater velocity acquired compensates. Mathematically we see from (76) that $\dot{x} = -x_0\sqrt{k}\,\sin\sqrt{k}\,t$. This shows that, though the velocity is 0 at $t = 0$, as t increases the velocity depends on x_0.
2. The motion is given by (76) with $x_0 = 100$ and k is the constant GM/R^3. Hence $x = 100\cos\sqrt{k}\,t$. Then $\dot{x} = -100\sqrt{k}\,\sin\sqrt{k}\,t$. The maximum velocity is the amplitude, namely, $100\sqrt{k}$.

3. Before we applied any initial conditions we derived (75). Now our initial conditions are $x = x_0/3$ when $t = 0$ and $\dot{x} = 0$ when $t = 0$. If we follow the procedure of fixing A and B that was used in the text, we obtain $x = (x_0/3) \cos\sqrt{k}t$. The period is still $2\pi/\sqrt{k}$. This last fact is interesting because the period is the same as when the object traverses the entire path UV.
4. If we start with (75) and apply the initial conditions $x = 0$ at $t = 0$ and $\dot{x} = 0$ at $t = 0$ we get in place of (76) just $x = 0$.
5. If we start with (75) and apply the initial conditions $x = \bar{x}$ when $t = 0$ and and $\dot{x} = 0$ when $t = 0$ we get $x = \bar{x} \cos\sqrt{k}t$ in place of (76). Thus for every value of $\bar{x}$ except 0, the period is $2\pi/\sqrt{k}$. For $\bar{x} = 0$ the period is 0 (as in Exercise 4). Hence the graph of period versus $\bar{x}$ is a line segment parallel to the $\bar{x}$ axis and extending from $-x_0$ to $+x_0$ but with no point on the line segment at $\bar{x} = 0$.
6. The physical phenomenon discussed here is best compared with the phenomenon of the bob on the spring. Equation (a) is the analogue of (50). Equation (b) is the analogue of (49). And the quantity (c) is the quantity $-ky$ below (50). The resulting differential equation is the analogue of (51) and is $100\ddot{y} = -32 \cdot 15.6\pi y$. Here when the cylinder is depressed below its equilibrium position the buoyant force of the water replaces the upward pull of the spring. When the cylinder rises above the equilibrium position, there is a downward force which is its weight in air minus the buoyant force on the portion of the cylinder still in the water. This is the analogue of the downward force exerted in the case of the spring, when the y-value is positive but less than d; the force is the weight of the bob minus the upward pull of the spring.

   Since the differential equation ($m\ddot{y}$ = net force) is $\ddot{y} = -4.992\pi y$, we can use (52) with k/m replaced by 4.992π. Then $y = A \sin\sqrt{4.992\pi}\,t + B \cos\sqrt{4.992\pi}\,t$. The initial conditions are that $y = -2$ when $t = 0$ and $\dot{y} = 0$ when $t = 0$. Then $y = -2 \cos\sqrt{4.992\pi}\,t$.
7. We start with the formula $T = 2\pi/\sqrt{k}$ where $k = GM/R^3$. Since the theory of the text does not restrict M and R to be the mass and radius of the earth we may let them be the mass and radius of the moon. Then, letting the letters M and R still stand for values of the earth's mass and radius, $T = 2\pi/\sqrt{(GM/81)/(3R/11)^3} = (54\sqrt{3}\,\pi/11\sqrt{11})(1/\sqrt{k})$.

   One may choose to let M and R stand for the mass and radius of the moon, in which case $GM = 5.3R^2$, and k still stands for GM/R^3 but M and R refer to the moon. Then $T = 2\pi/\sqrt{k} = 2\pi/\sqrt{GM/R^3} = 2\pi/\sqrt{5.3R^2/R^3} = 2\pi\sqrt{R}/\sqrt{5.3}$. This is also a correct answer. If in the next to the last expression one replaces 5.3 by 32/6, R by 3R/11 and then $32R^2$ by GM one obtains an answer in terms of the k of the text where M and R refer to values for the earth. This latter answer can be put in the form $T = 6\sqrt{2}\,\pi/\sqrt{11k}$. Here k has the value GM/R^3 where M and R stand for the earth's values. Of course the replacement of 5.3 by 32/6 is an approximation.

CHAPTER 10, SECTION 6, FOURTH SET

1. The function, if we use the more convenient form (80), is $y = a\sin[(2\pi/\tau)t + \phi]$. Then $\dot{y} = a(2\pi/\tau)\cos[(2\pi/\tau)t + \phi]$. The maximum value of $\dot{y}$ occurs when the cosine is 1 and then $\dot{y}$ is $2\pi a/\tau$.
2. From Exercise 1 we see that the maximum velocity is the amplitude a times $2\pi/\tau$, where τ is the period. Then $2 = 2\pi a/\tfrac{1}{5}$ and $a = 1/5\pi$.

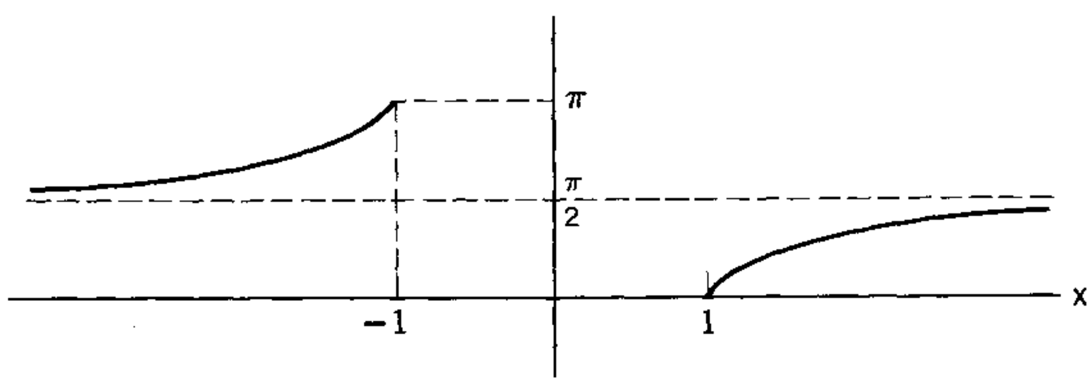
Solutions to Chapter 11

CHAPTER 11, SECTION 2

1. Solve for x.
2. (b) $-30°$; (d) $120°$; (f) $-17°27'$; (h) $180°$.
3. (b) $-\pi/2$.
4. (b) $\sqrt{3}/2$; (c) $2/\sqrt{3}$; (d) $\sqrt{3}/2$; (e) $\sqrt{3}/2$; (f) 0.5; (h) -1.

CHAPTER 11, SECTION 3

1. Start with $y = \cot x$. Then $dy/dx = -\csc^2 x$. Hence $dx/dy = -1/\csc^2 x = -1/(1+\cot^2 x) = -1/(1+y^2)$. Changing to x and y for independent and dependent variables, respectively, we have $dy/dx = -1/(1+x^2)$.
2. (a) Use (22) with $u = x/3$.
   (b) Use (23) with $u = x^2$. Then $y' = -2x/\sqrt{1-x^4}$.
   (c) Use (24) with $u = 1/x$.
   (d) Use (22) with $u = (x-1)/x$. We obtain first $y' = \{1/\sqrt{1-[(x-1)^2/x^2]}\}(1/x^2)$. In taking the x^2 out of the radical we must write $|x|$ because the value of the entire radical is positive. Since $1/x^2$ is also positive we must write $y' = (1/|x|)(1/\sqrt{2x-1})$.
   (e) First $f'(x) = 2\sin^{-1}x\, d(\sin^{-1})/dx$.
   (f) Use (24) with $u = \sqrt{x^2+1}$. Then $f'(x) = (x/\sqrt{1+x^2})[1/(2+x^2)]$.
   (g) Use (22) with $u = \cos x$. Then $y' = -\sin x/|\sin x|$.
   (h) Use (24) with $u = \sqrt{x-1}$. Then $f'(x) = 1/2x\sqrt{x-1}$.
   (i) Use (26) with $u = 3x$. Then $y' = \pm 1/3x\sqrt{9x^2-1}$.
   (j) Use (26) with $u = 2x-3$. Then $y' = \pm 1/\sqrt{3x-x^2-2}$.
   (k) Use (25) with $u = 3x^2$. Then $y' = -6x/(1+9x^4)$.
3. See Figure 1.



The slope is always positive.

4. For $0 \le x \le \pi/2$, $y = \pi/2 - x$; for $\pi/2 \le x \le 3\pi/2$, $y = x - \pi/2$; for $3\pi/2 \le x \le 2\pi$, $y = 5\pi/2 - x$.
5. In each case find d^2y/dx^2 and set it equal to 0 to find the abscissa of the point of inflection. Ans. to (b): $x = 0$.
6. $dy/dx = 1/(dx/dy)$. Now to differentiate with respect to x, regard dx/dy as a function of y with y as a function of x. Then $d^2y/dx^2 = [(-d^2x/dy^2)/(dx/dy)^2]\cdot(dy/dx)$. Since $dy/dx = 1/(dx/dy)$, we have the text result.
7. (a) $\tan^{-1}x/x = (\tan^{-1}x - \tan^{-1}0)/(x-0)$. Then $\lim_{x\to 0} \tan^{-1}x/x = d(\tan^{-1}x)/dx$ at $x = 0$. From (24) we see that the answer is 1.
   (b) The idea here is the same as in (a) except that we have the derivative of $\tan^{-1}x$ at $x = 1$. From (24) we see that the answer is $\frac{1}{2}$.
8. From Fig. 11-12 we see that $\theta = \tan^{-1}(y/a)$. Now $\dot{\theta} = d\theta/dt = (d\theta/dy)(dy/dt)$. From (24) we find that $d\theta/dy = a/(a^2+y^2) = \cos^2\theta/a$. Also $dy/dt = v$. Hence $\dot{\theta} = (v/a)\cos^2\theta$. Now $\ddot{\theta} = d\dot{\theta}/dt = (v/a)2\cos\theta(-\sin\theta)\dot{\theta} = -(2v^2/a^2)\cos^3\theta\sin\theta$.
9. $\theta = \tan^{-1}(gt^2/2a)$. Here θ is a function of t. To find $\dot{\theta}$ use (24) with $u = gt^2/2a$. Then the text answer follows at once and to find $\ddot{\theta}$ merely differentiate $\dot{\theta}$ with respect to t.

CHAPTER 11, SECTION 4

1. (a) Let $u = x^2$. Then we have the form (28).
   (b) Let $u = x^3$. Then we have the form (28).
   (c) Write $y' = (1/a)\{1/[1+(x/a)^2]\}$. Then we have the form (30) with $u = x/a$.
   (d) Write $y' = (1/a^2)\{1/[1+(bx/a)^2]\}$. Let $u = bx/a$. Then $du/dx = b/a$. Hence write $y' = (1/ab)\{1/[1+(bx/a)^2]\}(b/a)$. Hence $y = (1/ab)\tan^{-1}(bx/a) + C$.
   (e) Except for the factor $\frac{1}{2}$, the y' is in the form (32) [or (33)]. Hence $y = \frac{1}{2}\sec^{-1}x + C$.
   (f) Write $y' = (1/\sqrt{5})(1/\sqrt{1-(4x^2/5)})$. Then y' is in the form (28) with $u = 2x/\sqrt{5}$. Then $du/dx = 2/\sqrt{5}$. This constant factor can be introduced and the result is in the text.
   (g) Write $y' = \frac{1}{16}\{1/[1+(3x/4)^2]\}$. Then y' is in the form (30) with $u = 3x/4$, except for the factor $\frac{3}{4}$. The answer is in the text.
   (h) Write $y' = \frac{1}{4}(1/\sqrt{1-(3x/4)^2})$. Then y' is in the form (28) with $u = 3x/4$, except for the factor $\frac{3}{4}$. The answer is in the text.
   (i) y' is in the form (32) [or (33)] with $u = 2x$. Then $du/dx = 2$. Hence write $y' = \frac{1}{2}[1/(u\sqrt{u^2-1})](du/dx)$. Then $y = \frac{1}{2}\sec^{-1}2x + C$.
   (j) Let $u = x^3$ and write $f(x) = (1/3)\int 3x^2dx/\sqrt{1-(x^3)^2}$. Now use (28). $f(x) = (1/3)\sin^{-1}x^2 + C$.
   (k) $f(x) = \int x dx/(x^4+3) = (1/2)\int 2x dx/[(x^2)^2+3]$. Use (30). $f(x) = (\sqrt{3}/6)\tan^{-1}(x^2/\sqrt{3}) + C$.

(ℓ) Write the given integral as $(1/2)\int 2\sec x\tan x\,dx/\sqrt{3^2+(2\sec x)^2}$. Use (30). Then $y = (1/6)\tan^{-1}[2\sec(x/3)]+C$.

(m) Break up the given integral into $\int x\,dx/\sqrt{1-x^2} + 3\int dx/\sqrt{1-x^2}$. The first integral is evaluated by letting $u = 1-x^2$ and using the inverse of the power rule. The second integration uses (28). Then $y = -\sqrt{1-x^2}+3\sin^{-1}x+C$.

(n) Write the given integral as $dx/[(x+5)^2+5] = (1/5)\int dx/[((x+5)/\sqrt{5})^2+1]$. Now let $u = (x+5)/\sqrt{5}$ and use (30). Then $y = (\sqrt{5}/5)\tan^{-1}[(y+5)/\sqrt{5}]+C$.

2. By our formula for area, $dA/dx = y$ or $A = \int_{-\sqrt{3}}^{\sqrt{3}} dx/(9+x^2)$. Then with (30), $A = 1/3\tan^{-1}(x/3)$. Now substitute $\sqrt{3}$ and $-\sqrt{3}$ for x and subtract. Thus $A = 1/3[\tan^{-1}(\sqrt{3}/3)-\tan^{-1}(-\sqrt{3}/3)]$. But $\tan^{-1}(-\sqrt{3}/3) = -\tan^{-1}(\sqrt{3}/3)$. Hence $A = (1/3)2\tan^{-1}(\sqrt{3}/3)$. We can use the table to find that $\tan^{-1}(\sqrt{3}/3) = \pi/6$ or happen to remember this. Then $A = \pi/9$.

CHAPTER 11, SECTION 5

1. The steps parallel those of the text where the substitution $x = a\sin\theta$ was used. The two answers agree because $\sin^{-1}(x/a)$ and $\cos^{-1}(x/a)$ differ by a constant.
2. Our method of finding areas is to start with $dA/dx = y$. Here $y = (b/a)\sqrt{a^2-b^2}$. Hence we must integrate $(b/a)\sqrt{a^2-x^2}$. The integral in view of (53) is $(ab/2)\sin^{-1}(x/a) + (bx/2a)\sqrt{a^2-x^2}$. If we now substitute a for x and then 0 for x and subtract the second result from the first we get $\pi ab/4$.
3. Whenever a radical occurs a change of variable which eliminates the radical is usually helpful.
   (a) This can be done by factoring the 9 out of the radical and thus reducing to (28). But to use change of variable let $x = 3\sin\theta$. Then $dy/d\theta = (dy/dx)(dx/d\theta) = (1/\sqrt{9-9\sin^2\theta})\,3\cos\theta = 1$. Then $y = \theta + C$ or $y = \sin^{-1}(x/3) + C$.
   (b) Let $x = \frac{4}{3}\sin\theta$. Then $dy/d\theta = (dy/dx)(dx/d\theta) = \frac{1}{4}(1/\cos\theta)\,\frac{4}{3}\cos\theta$. Hence $y = \theta/3 + C = \frac{1}{3}\sin^{-1}(3x/4) + C$.
   (c) Let $x = \frac{4}{3}\tan\theta$ and follow the procedure of (a) or (b).
   (d) This is (48) with $a = 3$. Hence read off the answer from (52).

(e) If $x = a\tan\theta$, then $dy/d\theta = [1/(a^2 + a^2\tan^2\theta)^{3/2}]a\sec^2\theta = (1/a^2)\cos\theta$. Then $y = (1/a^2)\sin\theta + C$. If $\tan\theta = x/a$ then $\sin\theta = x/\sqrt{a^2 + x^2}$.

(f) Let $x = a\sin\theta$. Then $dy/d\theta = (1/a^2)\sec^2\theta$; $y = (1/a^2)\tan\theta$ $= x/a^2\sqrt{a^2 - x^2} + C$.

(g) Let $x = 4\sin\theta$. Then $dy/d\theta = 16\sin^2\theta$. Replace $\sin^2\theta$ by $(1 - \cos 2\theta)/2$. Then $y = 8\theta - 4\sin 2\theta$. Replace $\sin 2\theta$ by $2\sin\theta\cos\theta$ and transform back to x.

(h) Let $x = 5\sin\theta$. Then $y = \int\tan^2\theta d\theta = \int(\sec^2\theta - 1)d\theta = \tan\theta - \theta$. Now transform back to x.

(i) Let $x = 5\sin\theta$. Then $y = \int(\csc^2\theta - 1)d\theta$. Then $y = -\cot\theta - \theta$ Transforming back to x gives $y = (-\sqrt{25 - x^2}/x) - \sin^{-1}(x/5) + C$.

4. The procedure is the same as in Exercise 2. However instead of the end values a and 0 we have here 2 and 0.

CHAPTER 11, SECTION 6

1. Use (69) with $r_1 = R + 2000 \cdot 5280 = 3R/2$ and $r = R$. The calculation is extensive and the accuracy of the answer will depend on how many decimal places are carried. The text answer is approximate. A result between 1250 and 1300 seconds is good enough for present purposes. We could also use (70) and calculate the θ for which $R = (3R/2)\cos^2\theta$ and then substitute in (70).
2. Since an acceleration of 32 ft/sec^2 is greater than the actual gravitational acceleration the object will acquire greater velocity and take less time to fall 2000 miles. To calculate the time we use the method of Chapter 3. That is, $a = -32$, $v = -32t + C$; $C = 0$ because $v = 0$ when $t = 0$. Then $s = -16t^2 + C$ and if s is measured from the height of 2000 miles, $s = 3R/2$ when $t = 0$. Hence $s = -16t^2 + 3R/2$. Now calculate t when $s = R$. Then $t = \sqrt{R/32} = 812$ seconds.
3. The time of flight is the same as if the object were dropped from a height of 100,000 ft and traveled to the surface of the earth. Thus, as in Exercise 1, we can use (69) or (70). In either case $r_1 = 4000 \cdot 5280 + 100{,}000$ and r $4000 \cdot 5280$. Let us use (70). Then we must first use (65) to find the value of θ when $r = 4000 \cdot 5280$. This gives $\theta = 86° 20'$ approx. Use of (70) gives 98 sec. approx. Again the accuracy will depend on the number of decimal places carried.
4. As in Exercise 3 the time required is the same as if an object falls from a height of 240,000 miles to the surface of the earth. We can use (69) or (70). In either case $r_1 = 240{,}000$ miles $= 60R$ and $r = R$. The arithmetic is again lengthy and the accuracy depends on the number of decimal places carried.
5. Following the suggestion we start with $v^2/2 = v_0^2/2 + GM/r - GM/R$. Here v_0 is the velocity at the surface of the earth and v is the velocity at the distance r from the center of the earth. We let v_0 be the escape velocity $\sqrt{2GM/R}$. Then $v^2/2 = GM/r$ and $v = \sqrt{2GM/r}$. Then since $v = dr/dt$, $dt/dr = r^{1/2}/\sqrt{2GM}$ and $t = \frac{2}{3}r^{3/2}/\sqrt{2GM} + C$. If we let $t = 0$ when $r = R$, $C = -2R^{3/2}/3\sqrt{2GM}$ and so $t = 2r^{3/2}/3\sqrt{2GM} - 2R^{3/2}/3\sqrt{2GM}$.

6. The force of attraction acting say on the sphere to the right is, according to the law of gravitation, wherein now $M = m$ and the distance between the centers of the two spheres is $2x$, is $-Gm^2/4x^2$. Then $m\ddot{x} = -Gm^2/4x^2$ or $\ddot{x} = -Gm/4x^2$. Then, following the suggestion, we take over the theory of the text, starting with (62), but replace GM by $Gm/4$. We may then go directly to (69) and replace GM there by $Gm/4$. In our case $r_1 = 3$ and $r = 1$ because the spheres are in contact when each is 1 foot away from the origin. The answer of two hours is approximate because the accuracy of the answer depends on the number of decimal places carried.
7. (a) We cannot use (69) because letting r_1 be infinite does not give a clear value for t. We start with $a = d^2r/dt^2 = -GM/r^2$ and write $dv/dt = -GM/r^2$. Then as in Chap. 7, sec. 8, $v\,dv/dr = -GM/r^2$. Integrating gives $v^2/2 = GM/r + C$. Now our initial condition is that $v = 0$ when r becomes infinite. Hence $C = 0$ and $v^2/2 = GM/r$. Then $v = -\sqrt{GM/r}$; the minus sign enters because v is negative. Since $v = dr/dt$, $dt/dr = -r^{1/2}/\sqrt{2GM}$. Then $t = -\frac{2}{3}r^{3/2}/\sqrt{2GM} + C$. If we agree to measure time from the instant the body reaches the center, then $t = 0$ when $r = 0$ and $C = 0$. Then $r = (-3/2)^{2/3}(2GM)^{1/3}t^{2/3}$. We must understand t to be negative as the formula for t shows. That is, by taking the acceleration negative and then the velocity negative, r must decrease with increasing t. Since t is 0 when the object reaches the center, it is negative previously.
   (b) We see from the expression for t that when r is infinite (and positive) t is $-\infty$. Hence it takes an infinite time to reach the center.
   (c) We see from the expression for v that v is infinite when r is 0.
   (d) Yes. The factor $(-3/2)^{2/3}$ is positive. For positive t values r is positive.
   (e) Since r increases with t, starting from $r = 0$ for $t = 0$, the object would have to move away from the center in the direction of increasing r as t increases. This would not happen physically because the body never reaches the center. Hence in the present situation the formula has no physical meaning for positive t.

Solutions to Chapter 12

CHAPTER 12, SECTION 2,

1. (b) 2.8451; (d) $\bar{2}.8451$
2. (b) 2.5353; (d) 2.7535; (e) $\log 21 = \log 7 + \log 3$, $\log 3 = \frac{1}{2}\log 9$. Ans. 1.3222.

CHAPTER 12, SECTION 3 (*p.*)

1. (a) If $x = \log_e 3$, then $e^x = 3$. Since e is about 2.7, x is somewhat greater than 1.
   (b) $\log_e 9 = 2 \log_e 3$; by (a), the integral part is at least 2.
   (c) Since $3^4 = 81$, $e^4 < 81$; hence 4 is a good estimate.
   (d) Log of 1 to any base is 0.
2. (b) 4; (c) 0.
3. (b) 2.94444; (d) $7.697 - 10$; (e) $9.307 - 10 = -0.6931$.
4. (a) 0.600; (c) 30.
5. Compare Figs. 12-1 and 12-2.
6. (a) The y-values of $3 \log x$ are three times as large as those of $\log x$.
   (b) Add 2 to each y value of $y = \log x$.
   (c) The curve of $y = \log (x + 3)$ is the curve of $y = \log x$ translated 3 units to the left.
   (d) The curve is that of $\log x$ translated 2 units to the left and with each y-value then multiplied by 3.
   (e) $\operatorname{Log}(-x)$ is the reflection in the y-axis of $\log x$. The curve of $y = \log (5 - x)$ is the curve of $y = \log(-x)$ translated 5 units to the right.
7. Let $e^{\ell} = a^m = x$. Then $\log_e e^{\ell} = \log_e a^m = m \log_e a = \log_a x \log_e a$. Hence $\log_e x = \log_a x \log_e a$. Then if $y = \log_a x$, $y = (1/\log_e a)\log_e x$. Hence $y' = 1/x \log_e a$.
8. (a) Let $u = x^2$ and use the chain rule (20).
   (b) Let $u = \log x$. Then $y' = 2(\log x)/x$.
   (c) Let $u = x/(1 + x)$. Then $y' = (1/u)\, du/dx = [(1 + x)/x][1/(1 + x)^2]$.
   (d) Let $u = \sin x$. Then $y' = (1/\sin x) \cos x = \cot x$.
   (e) Let $u = x^2 + 3x$. Then $y' = [1/(x^2 + 3x)](2x + 3)$.
   (f) Differentiate as a product. $y' = \log x + 1$.
   (g) In view of (f), $y' = \log x$.
   (h) Differentiate as a quotient. Hence $y' = (1 - \log x)/x^2$.
   (i) Let $u = \log x$. Then $y' = (1/\log x)(1/x)$.
9. Yes. We now know that the integral of $\log x$ is $x \log x - x$.

10. (a) Keeping aside the factor $\frac{1}{3}$ we can apply (19).
(b) Let $u = x + 2$. Then $du/dx = 1$. Hence apply (21). $y = \log(x + 2) + C$.
(c) Let $u = x^2 + 1$. Then $du/dx = 2x$. Hence apply (21).
(d) Let $u = x^2 + 1$. Then $du/dx = 2x$. Write $y' = \frac{1}{2}[1/(x^2 + 1)]2x$. Apply (21). Then $y = \frac{1}{2}\log(x^2 + 1) + C$.
(e) Let $u = x^2 - 6x + 10$. Then $du/dx = 2x - 6$. Write $y' = \frac{1}{2}[1/(x^2 - 6x + 10)](2x - 6)$. Apply (21).
(f) Let $u = \sin x$. Then $du/dx = \cos x$. Apply (21) or more generally (31).
(g) Let $u = \cos x$. Then $du/dx = -\sin x$. Write $y' = -(1/\cos x)(-\sin x)$. Apply (21) or more generally (31).
(h) See (g).
(i) See (f).
(j) Let $u = 1 - \cos x$. Then $du/dx = \sin x$. Hence by (21) $y = \log(1 - \cos x) + C$.
(k) Let $u = \tan x$. Then $du/dx = \sec^2 x$. Apply (21) or (31).
(l) Let $u = \log x$. Then $du/dx = 1/x$. Write $y' = (1/\log x)(1/x)$. $y = \log(\log x) + C$.

11. (a) Let $u = x^2+2$. Then $du/dx = 2x$. Write the given integral as $3/2\int 2x dx/(x^2+2)$. Hence $(3/2)\log(x^2+2)+C$.
(b) Let $u = 1-x^3$. Multiply the numerator by -3 and divide outside the integral sign by -1/3. Since $du/dx = -3x^2$, then the answer is $(-1/3)\log|1-x^3|+C$. The absolute value takes care of values of x for which $1-x^3 > 0$ and $1-x^3 < 0$.
(c) Let $u = x^2+2x+5$. Then $du/dx = 2x+2$. Multiply the numerator of the given integral by 2 and divide outside by 1/2. Hence $(1/2)\log(x^2+2x+5)+C$.
(d) Break up into a difference of integrals. Let $u = 2x-5$ in the first and $u = 2x+3$ in the second. Multiply numerators by 2 and divide outside by 1/2. Then $(1/2)\log|2x-5|-(1/2)\log|2x+3|+C = \sqrt{\left|\frac{2x-5}{2x+3}\right|}+C$.
(e) $\int\tan(x/2)dx = \int\sin(x/2)dx/\cos(x/2)$. Let $u = \cos(x/2)$. Then $du/dx = -(1/2)\sin(x/2)$. Multiply numerator of integral by -1/2 and outside by -2. Hence $-2\log|\cos(x/2)|+C = 2\log|\sec(x/2)|+C$.

12. $A = \int_1^{10} dx/x = \log x\Big|_1^{10} = 2.3026-0 = 2.3026$.

13. $dR/ds = k/S$. $R = k\log S+C$.

14. For $n = 1$, $d(\log x)/dx = 1/x$. Since $0!$ is defined to be 1, the theorem holds for $n = 1$. Now assume for $n = k$ that $d^k(\log x)/dx^k = (-1)^{k-1}(k-1)!/x^k$. Then $d^{k+1}(\log x)/dx^k = (-1)^{k-1}(k-1)!(-k)x^{-k-1}$. We see that if the theorem is true for $n = k$ it is true for $n = k + 1$. Since it is true for $n = 1$ it is true for all positive integral n.

15. $\lim_{x\to 1}\log x/(x-1) = \lim_{x\to 1}(\log x - \log 1)/(x-1)$. This is the expression for the derivative of $\log x$ at $x = 1$, or $y' = 1/x$ at $x = 1$ or 1.

CHAPTER 12, SECTION 2, FIRST SET

1. (a) $y = 3e^x$ has y-values 3 times as large as $y = e^x$.
   (b) $y = e^{-x}$ is the reflection in the y-axis of $y = e^x$.
   (c) The curve of $y = e^{(x+2)}$ is the curve of $y = e^x$ displaced two units to the left.
   (d) The graph of $y = e^x + 2$ is 2 units higher than that of $y = e^x$.
   (e) Add ordinates of $y = x$ and $y = e^x$.
   (f) $y = e^{3x}$ is steeper than $y = e^x$.
   (g) Compare (f) and then reflect in the y-axis.

CHAPTER 12, SECTION 4, SECOND SET

1. (a) Let $u = x^2$ and use (38). (b) Let $u = -2x$ and use (38). Then $y' = -2e^{-2x}$.
   (c) Let $u = -1/x$ and use (38). (d) Let $u = \sin x$; then $y' = e^{\sin x}\cos x$.
   (e) Differentiate as a product; $y' = e^x/x + e^x \log x$.
   (f) Differentiate as a product.
   (g) Differentiate as a product; then $y' = -e^{-x}(\cos 2x + 2\sin 2x)$.
   (h) Let $u = e^x$.
   (i) Differentiate as a quotient. Then $y' = 2/(e^x + e^{-x})^2$.
   (j) Let $u = e^x/(1+e^x)$. Then $y' = (1/u)(du/dx) = [(1+e^x)/e^x][e^x/(1+e^x)^2]$ $= 1/(1+e^x)$.
2. The slope is, of course, y' at $x = 0$. Hence 1.
3. Use the e-table to make a table of values.
4. (a) Let $u = -x$. Write $y' = -e^{-x}(-1)$ and use (40).
   (b) Let $u = x^2$. Write $y' = \frac{1}{2}e^{x^2}\cdot 2x$ and use (40). Then $y = e^{x^2}/2 + C$.
   (c) Let $u = \sin x$. Apply (40).
   (d) Let $u = -1/x$. Then $du/dx = 1/x^2$. Apply (40). Ans. $y = e^{-1/x} + C$.
   (e) Let $u = -x/2$. Then $du/dx = -1/2$. Write $f'(x) = -2e^{-x/2}(-1/2)$. Hence $f(x) = -2e^{-x/2}+C$.
5. (a) Let $u = e^x + e^{-x}$. Then the numerator is du/dx and we have the form $y' = (1/u)(du\ dx)$. Hence $y = \log u + C$.
   (b) Multiply numerator and denominator by e^{-x} and the problem reduces to (a).
   (c) Let $u = 1 + e^{2x}$. Then $du/dx = 2e^{2x}$. Write $y' = \frac{1}{2}2e^{2x}/(1+e^{2x})$ and we have the form $y' = \frac{1}{2}(1/u)(du/dx)$.
6. (a) Let $u = 4x$. $\int e^{4x}dx = (1/4)\int e^{4x}4dx = (1/4)e^{4x}+C$.
   (b) Let $u = 1/x^2$. Then $du/dx = -2/x^3$. Now $\int e^{1/x^2}dx/x^3 =$ $(-1/2)\int e^{1/x^2}(-2/x^3)dx = -1/2e^{1/x^2}+C$
   (c) Let $u = -x^2+3$. Then $du/dx = -2x$. $\int e^{-x^2+3}xdx = -(1/2)\int e^{-x^2+3}$ $(-2x)dx = -(1/2)e^{-x^2+3}+C$.

(d) $(e^x+2)^2 = e^{2x}+4e^x+4$. Integrate each term separately. Answer is $(1/2)e^{2x}+4e^x+4x+C$.

(e) Let $u = e^x+1$. Then $du/dx = e^x$. The given integral is $\int u^4(du/dx)dx = (u^5/5)+C = (e^x+1)^5/5+C$.

(f) Let $u = e^{2x+5}$. Then $du/dx = 2e^{2x}$. Hence the given integral is $1/2\int(1/u)(du/dx)dx = (1/2)\log u+C = (1/2)\log(e^{2x}+5)+C$.

7. $A = \int_0^1 xe^{x^2}dx = (1/2)\int_0^1 e^{x^2}2xdx = (1/2)e^{x^2}\Big|_0^1 = (1/2)e-1/2$.

8. (a) $F(2) = \int_0^2 e^{-x^2}dx$. Since the curve lies above the x-axis, the area is positive.

(b) $F(-2) = \int_0^{-2} e^{-x^2}dx = -\int_{-2}^0 e^{-x^2}dx$. This area is the same as $F(2)$ except for the minus sign. See (a) for $F(2)$.

(c) $F(3) = \int_0^3 e^{-x^2}dx$. This covers more area than $F(2)$.

(d) As t increases $F(t)$ covers more area to the right of the y-axis. Since the area is always above the x-axis, $F(t)$ increases.

9. The graph is the reflection of $y = e^x$ in the y-axis.

10. Graph e^{-x} and $\sin x$ on the same set of axes and choose a number of x-values. At these multiply (roughly) the values of e^{-x} and $\sin x$.

11. At $x = 0, h, 2h, \ldots,$ $y = 1, e^{kh}, e^{k2h}, \ldots$. We see that the y-values are in geometric progression with common ratio e^{kh}. If the x-values are $a, a+h, a+2h, \ldots$, then the common ratio is still e^{kh}.

12. $\lim_{h\to 0}(e^h-1)/h = \lim_{h\to 0}(e^h-1)/(h-0)$. This is $\lim_{\Delta x\to 0}\Delta y/\Delta x$ for the function e^x at $x = 0$. Hence the answer is the derivative of e^x at $x = 0$ or 1.

13. $y' = kDe^{kx} = ky$.

14. The average cost $A = 5e^{x/8}/x$. Then $A' = [x(5/8)e^{x/8}-5e^{x/8}]/x^2$. $e^{x/8}(5x/8-5)/x^2$. $A' = 0$ when $x = 8$. Also A' changes from - to +. Hence $x = 8$ is a minimum.

CHAPTER 12, SECTION 5

1. Let T be the temperature of the object. Then Newton's law states that $\dot{T} = -k(T-T_0)$, the minus entering, when k is positive, because T decreases with increasing time and so $\dot{T}$ is negative. Then $dt/dT = -1/k(T-T_0)$. Let $u = T-T_0$. Then $t = (-1/k)\log(T-T_0)+C$. Then $\log(T-T_0) = -kt+kC$ or $T = T_0+e^{-kt+kC}$. Since kC is also arbitrary, $T = T_0+De^{-kt}$. In our problem when $t = 0$, $T = 100$. Hence $D = 100-T_0$. Also $k = 0.01$. Then $T = T_0+(100-T_0)e^{-.01t}$.

2. Here we argue that at the end of any one minute the temperature is .99 times that at the beginning of the minute. Then $T = 100(0.99)^t$. The problem does not call for continuous change at each instant.
3. Here this is continuous change but the rate is not proportional to the existing temperature. Rather we are given the change as a fixed quantity. In t minutes the loss is $0.01t$. Hence $T = 100 - 0.01t$.
4. We know merely the net change at the end of each year. Then, as in Exercise 2, the population at the end of any year is 1.03466 times the population at the beginning of the year. If P_0 is the population when $t = 0$, then $P = P_0(1.03466)^t$. Presumably the population increases continuously at some rate r per year of the existing population at any instant. If we knew r, then doing the problem as one of continuous rate of change proportional to the existing population we would have $\dot{P} = rP$ and as our solution $P = P_0e^{rt}$. The e^r is the 1.03466. The problem is analogous to continuous compounding to money at the rate of 0.04 per year. The net rate per year as pointed out in (46) is $e^{0.04}$.
5. Use (54). In our problem we know that when $t = 0$, $P = 5000$. Hence $D = 5000$. Also when $t = 20$, $P = 15000$. Then $15{,}000 = 5000e^{20k}$. As in the text following (54) we determine k. Taking logarithms of both sides, $20k = \log_e 3 = 1.09861$. Hence $k = .055 = .06$. This gives the text's answer.
6. We take over from Exercise 1, that $T = T_0 + De^{-kt}$. In our case when $t = 0$, $T = 100$. Hence $D = 100 - T_0$. Our T_0 is $40°$. Hence so far $T = 40 + 60e^{-kt}$. We know that when $t = 2$, $T = 80°$. Hence $80 = 40 + 60e^{-2k}$. Then $-2k = \log \frac{2}{3} = 9.594 - 10 = -.406$ and $k = .204$. Hence $T = 40 + 60e^{-0.204t}$. To find t when $T = 43°$, we have $43 = 40 + 60e^{-0.204t}$ or $-0.204t = \log(.05) = 7.004 - 10 = -2.996$. Then $t = 14.6$ minutes.
7. Let S be the amount of undissolved sugar at any time t. Then $\dot{S} = -kS$, the minus entering because S is decreasing (and k is taken positive). Then, as in going from (47) to (49), $S = De^{-kt}$. When $t = 0$, $S = 200$. Hence $D = 200$ and $S = 200e^{-kt}$. When $t = 2$, $S = 200 - 100$ or 100. Then $100 = 200e^{-2k}$. Taking logarithms and solving k gives $k = .347$. Then $S = 200\,e^{-0.347t}$. To find t when $S = 200 - 150$, we have $50 = 200e^{-0.347t}$. Since $\log .25 = 8.614 - 10 = -1.386$, $t = 4$ minutes. From the formula for S, we see that when $S = 0$, t must be infinite. Formally, $-0.347t = \log 0 = -\infty$.
8. If the population is P_0, the increase after one year is P_0r and the population is $P_0 + P_0r$. After 2 years the population is $P_0 + P_0r + r(P_0 + P_0r) = (P_0 + P_0r)(1 + r) = P_0(1 + r)^2$. After t years, $P = P_0(1 + r)^t$. If the population increases continuously then after t years $P = P_0e^{kt}$. Then $(1 + r)^t = e^{kt}$ or $k = \log(1 + r)$.
9. As in (54) $N = De^{kt}$. When $t = 0$, $N = 100$. Hence $D = 100$. We are given that $k = 0.2$. Hence the text's answer.

10. Let I denote intensity and d, depth. Then $I' = dI/dd$. We have that $\dot{I} = -kI$ and so $I = De^{-kd}$ We know that $D = I_0$ = intensity when $d = 0$. We are told that when $d = 10$, $I/I_0 = \frac{1}{2}$. Hence $\frac{1}{2} = e^{-10k}$ or $-10k = \log \frac{1}{2} = 9.307 - 10$ $= -0.693$. $k = 0.693$. If we take $\dot{I} = kI$ to start with then $k = -0.0693$.
11. From $\dot{N} = -kN$, we have $N = De^{-kt}$. When $t = 0$, $N = 10^{12}$. Hence $D = 10^{12}$. Hence, so far, $N = 10^{12}e^{-kt}$. When $t = 4.5 \cdot 10^9$, $N = \frac{1}{2}10^{12}$. Hence $-4.5 \cdot 10^9 k$ $= \log \frac{1}{2} = -0.693$ or $k = 1.6 \cdot 10^{-10}$.
12. Since $U = u(1 - e^{-\lambda t})$ and $u = P(1 - e^{-\lambda t})$, the net amount of uranium at any time t is $u - U = P(1 - e^{-\lambda t})e^{-\lambda t}$. We have that $P = 3400$ and we are to determine whether there is a value of t for which $u - U = 800$. Let $e^{-\lambda t} = z$. Then $800 = 3400(z - z^2)$ or $17z^2 - 17z + 4 = 0$. Then $z = (17 \pm \sqrt{17})/34$. Now $z = e^{-\lambda t}$. Hence $-\lambda t = \log[(17 \pm \sqrt{17})/34]$. We know that $\lambda = \frac{1}{10} \log 2 = .0693$. It is not necessary to get the precise values of $\log[(17 + \sqrt{17})/34]$ and $\log[(17 - \sqrt{17})/34]$. We need merely note that both numbers are positive and < 1. Hence they have negative logarithms and so there are two values of t at which $u - U = 800$. Moreover the function $u - U$ is of the form $Pz - Pz^2$. The two values of t for which $u - U = 800$ correspond to two values of z at which $u - U = 800$. In between these the curve has a maximum and so $u - U$ rises above 800. Hence disaster.
13. Let Q be the number of pounds of salt in the tank at time t. The rate of increase of salt at any time t, which is due to the inflow is 2 lbs/min. Since Q is the amount of salt present at any time t, $Q/100$ is the salt per gallon present because there are always 100 gallons present. The rate of outflow of salt is $2Q/100$ because 2 gallons of the mixture constantly flow out. Then $dQ/dt = \dot{Q} = 2 - 2Q/100 = (100 - Q)/50$. Then $dt/dQ = 50/(100 - Q)$. To integrate let $u = 100 - Q$. Hence $t = -50 \log(100 - Q) + C$ and $Q = 100 - De^{-t/50}$. When $t = 0$, $Q = 150$. Hence $D = -50$ and $Q = 100 + 50e^{-t/50}$. When $t = 1$ hr $= 60$ min., $Q = 100 + 50e^{-6/5}$. To calculate $e^{-1.2}$ use the e-table. $e^{-1.2}$ $= e^{-1} \cdot e^{-.2} = (.368)(.819) = .3$. Then $Q = 115$ lbs.
14. Let y be the number of gallons of impurities or impure quantity present at time t. Since there are 10 gals of mixture in the tank at any time t then $y/10$ is the amount of impure quantity present per gallon of water. Since 10 gal/hr flow out constantly the rate of change of impure water is $10(y/10)$ and this is $\dot{y}$. However, $\dot{y}$ is negative because y is decreasing. Hence $\dot{y} = -y$. Then $y = y_0e^{-t}$ where y_0 is the number of gallons of impurities present at $t = 0$. We are asked to find t when $y = y_0/2$. Then $\frac{1}{2} = e^{-t}$ and $-t = \log \frac{1}{2}$ $= -0.693$ or $t = 0.7$ hr.
15. We are asked to find t when $N(t) = N(0)/2$. Then $\frac{1}{2} = e^{-kt}$. Hence $-kt = \log \frac{1}{2}$ $= -0.693$ and $t = 0.693/k$.
16. (a) We are to find $N(t)$ when $t = 100$. $N(100) = N(0)e^{-0.0433}$. Use of the e-table gives $0.958\ N(0)$.
    (b) We have to find when $N(t) = N(0)/2$. Then $\frac{1}{2} = e^{-.000433t}$. Hence $-.000433t$ $= \log \frac{1}{2} = -0.693$. $t = 1600$ years.

17. $S = -sS$. Then $S = S_o e^{-st}$, where t is the time in years and s is the rate. When $S(t) = (1/2)S_o$, $e^{-st} = 1/2$; $-st = \log_e(1/2) = -0.693$. Hence $t = 0.693/s$.
18. We have that $dP/dt = 0.02P$. Hence $P = De^{0.02t}$. When $t = 0(1970)$, $D = 3$ billion. Then $P = 3e^{0.02t}$. When $P = 20$, $20 = 3e^{0.02t}$. Then $.02t = \log(20/3) = 2.4 - 1.1 = 1.3$. Then $t = 65$ years.
19. $dP/dt = k(20-P)$ where k is positive. Then $dt/dP = (1/k)[1/(20-P)] = (-1/k)[1/(20-P)](-1)$. Integration by letting $u = 20-P$ gives $t = -(1/k)\log(20-P)+C$. Then $P = 20-De^{-kt}$.
20. As in earlier exercises (see #15), $C = C_o e^{-kt}$ where $k = \log 2/5570$. Next $C = (1/10)C_o$ so that $1/10 = e^{-(\log 2/5570)t}$. Then $t = 5570 \log 10/\log 2 = 5570\ 2.303\ /0.693 = 18{,}500$ years.
21. By the method used many times (see for example (54)) we obtain $P(t) = De^{-(c/w)t}$ and since $P = P_o$ when $t = 0$, $P = P_o e^{-(c/w)t}$.
22. $y' = [A/c_2-c_1)](c_2 e^{-c_2 t} - c_1 e^{-c_1 t})$. At $y' = 0$, $e^{(c_1-c_2)t} = c_1/c_2$ so that $t = [1/(c_1-c_2)]\log(c_1/c_2)$.

CHAPTER 12, SECTION 6, FIRST SET

1. According to Newton's second law $m\dot{v} = -Kv^2$ or $\dot{v} = -kv^2$. Then $dt/dv = -(1/k)(1/v^2)$. Then $t = 1/kv + C$. When $t = 0$, $v = 100$. Hence $C = -1/100k$. Then $v = 100/(1 + 100kt)$.
2. Integrate with respect to t the result of Exercise 1. Let $u = 1 + 100kt$. Write $v = dx/dt = (1/k)[1/(1 + 100kt)](100k)$. Then $x = (1/k)\log(1 + 100kt) + C$. If $x = 0$ when $t = 0$, then $C = 0$. This gives $x = (1/k)\log(1 + 100kt)$.
3. When t becomes large, the velocity in the case of Exercise 1 becomes smaller and smaller. However if we look at (61) this is also true when the resistance is proportional to the velocity. However when the velocity becomes less than 1, the resistance in one case is kv^2 and in the other kv. Then the resistance is less in the former case and for very small v the resistance is much smaller in the case of kv^2 than in the case of kv. Hence the object loses less velocity and travels much farther.
4. (a) $\dot{v} = -k\sqrt{v}$. Then $dt/dv = (-1/k)(1/\sqrt{v})$. $t = (-2/k)\sqrt{v} + C$. When $t = 0$, $v = V$. Hence $C = 2\sqrt{V}/k$. Then $v = (\sqrt{V} - kt/2)^2$.
   (b) To integrate the expression for v, it is easier to let $u = \sqrt{V} - kt/2$. Then $x = (-2/3k)(\sqrt{V} - kt/2)^3 + C$. When $t = 0$, $x = 0$. Hence $C = (2/3k)V^{3/2}$. Hence $x = (-2/3k)(\sqrt{V} - kt/2)^3 + (2/3k)V^{3/2}$.
   (c) Take the value of the parentheses from (a) and substitute in the answer to (b). Then $x = 2(V^{3/2} - v^{3/2})/3k$.
   (d) The particle moves until $v = 0$. We see from (c) that then $x = 2V^{3/2}/3k$. If we use the answer to (b) we get ∞ for x but physically this cannot be correct because the particle does reach 0 velocity and there is no force to give it an acceleration at that point. The mathematics fails to represent the physics after v becomes 0.

5. This is mathematically the same as in the text from (60) to (65) except that the 100 there is now v_0.
6. In place of $\dot{v} = -kv$ we have $v\,dv/dx = -kv$. Then $dv/dx = -k$ and $v = -kx + C$. When $x = 0$, $v = v_0$. Hence $C = v_0$ and so $v = v_0 - kx$.
7. We may take over the result of Exercise 1 with 100 replaced by V. Then $v = V/(1 + Vkt)$. Likewise from Exercise 2, $x = (1/k)\log(1 + Vkt)$. In our case $k = 0.0068$. We find first the time in which v becomes $V/2$. This occurs when $t = 1/Vk$. If we substitute this in the formula for x we obtain $x = (1/k)\log(1 + 1)$. Since $\log 2 = .693$ and $k = 0.0068$, $x = 101.9$ ft.
8. In place of $\dot{v} = -kv^2$ we have $v\,dv/dx = -kv^2$ or $dv/dx = -kv$. Then by the usual method of working with dx/dv we find $v = Ve^{-kx}$. Now let $v = V/2$. Then $\frac{1}{2} = e^{-kx}$ or $e^{kx} = 2$ and $kx = \log 2$ so that $x = (\log 2)/k$. Since $k = .0068$, $x = 101.9$ ft.

CHAPTER 12, SECTION 6, SECOND SET

1. If we take k negative then $\dot{v} = -kv-32$. Hence $dt/dv = -1/(kv+32)$ and $t = -\int dv/(kv+32) = -1/k\int kdt/(kv+32) = -(1/k)\log(kv+32)+C$. Then $kv+32 = De^{-kt}$ or $v = (D/k)e^{-kt}-32/k$. When $t = 0$, $v = 0$, so that $D = 32$.
2. In the vacuum case $v = 32t$.
3. Use (68) and the fact that at $t = 0$, $v = 1000$. Hence $1000 = 32/k - D/k$. Then $D = 32 - 1000k$. If we substitute this in (68) we have $v = 32/k - (32 - 1000k)e^{-kt}/k = 32/k - (32/k - 1000)e^{-kt}$.
4. (a) The procedure is exactly as in Exercise 3 except that V replaces 1000. Hence $v = 32/k - (32/k - V)e^{-kt}$. If $V > 32/k$ the parenthesis is negative. Then as t increases, because e^{-kt} decreases, v decreases. If $V = 32/k$, the velocity v is constant and $32/k$. If $V < 32/k$, then v increases as t increases because less is subtracted.

   (b) The terminal velocity is $32/k$ in each case. We know that the basic differential equation is $\dot{v} = 32 - kv$. Since to start with $v = V$, if $V > 32/k$ the acceleration is negative and this remains so until $v = 32/k$. If $V = 32/k$, there is no acceleration at the start and then v continues at the value $32/k$ so that again there is no acceleration. If the initial V is $< 32/k$, then the initial acceleration is positive and the velocity increases. This keeps happening until $v = 32/k$ (at $t = \infty$).
5. From Exercise 3 we have $v = 32/k - (32/k - 1000)e^{-kt}$. Then $y = (32/k)t + (1/k)(32/k - 1000)e^{-kt} + C$. If we agree that $y = 0$ when $t = 0$ then $C = (-32/k^2) + 1000/k$. Then $y = (32/k)t + (1/k)(32/k - 1000)(e^{-kt} - 1)$.
6. The terminal velocity is always $32/k$ (see Exercise 4). Hence $32/k = 12$ and $k = 8/3$. With this value of k we may use (69) and (71) to compute v and y respectively at $t = 2$. From (69) $v = 12(1 - e^{-16/3})$ and $y = 12.2 + (9/2)e^{-16/3} - 9/2$. The values are $v = 11.94$ ft/sec and $y = 19.53$ ft.

7 . In falling freely $v = 32t$. Hence after 10 seconds the aviator's velocity is 320 ft. This is his initial velocity when the motion under air resistance begins. We may then use the result of Exercise 4 (a) with $V = 320$. We obtain $v = 32/k - (32/k - 320)e^{-kt}$.

8 . We have to start with that $v = v_0 + ky$. Then $\dot{y} = v_0 + ky$. Integrating by working with dt/dy, and since $y = 0$ when $t = 0$, $y = (v_0/k)(e^{kt} - 1)$. Now we can differentiate y to get $v = v_0 e^{kt}$ and $a = \dot{v} = kv_0 e^{kt}$.

9 . The assumption that $v = ky$ is a special case of Exercise (8) in which $v_0 = 0$. Then according to the result in (16) $y = 0$ for all t. The body does not fall.

10 . Here we start with $a = ky$. Now $a = \dot{v} = v\, dv/dy$. Hence $v\, dv/dy = ky$ or $v^2/2 = (ky^2/2) + C$. Since $v = v_0$ when $y = 0$, then $C = v_0^2/2$. Then $v^2 = ky^2 + v_0^2$ or $\dot{y} = \sqrt{ky^2 + v_0^2}$. Then $dt/dy = 1/\sqrt{ky^2 + v_0^2}$. Now use formula No. 38 in the integral tables (or derive it by a change of variable). Then, with $u = \sqrt{k}y$, $t = (1/\sqrt{k}) \log(\sqrt{k}\,y + \sqrt{ky^2 + v_0^2}) + C$ or $\sqrt{k}\,y + \sqrt{ky^2 + v_0^2} = De^{\sqrt{k}t}$. Since $y = 0$ when $t = 0$, $D = v_0$ and $\sqrt{k}\,y + \sqrt{ky^2 + v_0^2} = v_0 e^{\sqrt{k}t}$. To solve for y, write $\sqrt{ky^2 + v_0^2} = v_0 e^{\sqrt{k}t} - \sqrt{k}\,y$ and square and simplify.

CHAPTER 12, SECTION 7

1. At this point $v = 0$ and this occurs at the maximum height.
2. The type of problem which neglects air resistance was treated in Chap. 3. In this case $a = -32$ and $v = -32t + 1000$.
3. We know that $\dot{y}$ or v is 0 at maximum height. Hence use (75) and solve for t when $v = 0$. The algebra is straightforward and the answer is in the text.
4. Since (77) gives the height at any time t we substitute the value of t_1 obtained in Exercise 3 for t in (77). There is no difficulty in the algebra. One uses the fact that $e^{-\log z} = 1/e^{\log z} = 1/z$. The answer is in the text.
5. The formulas here, as in Chap. 3, are $v = 1000 - 32t$, $y = 1000t - 16t^2$. When $v = 0$, $t = {}^{125}/_4$ sec. and $y = 15{,}625$ ft.
6. We must calculate t_1 in Exercise 3 with $k = 0.5$. Then $t_1 = 2 \log(1 + {}^{125}/_8) = 2(2.777) = 5.554$ sec. Now we calculate y_1 in Exercise 4 with $k = 0.5$. Then $y_1 = 2000 - 128 \log(1 + {}^{125}/_8) = 2000 - 128(2.777) = 2000 - 355 = 1645$ ft.
7. We see from the comparison with Exercise 5 that the object reaches the maximum height very much faster but does not go nearly as high.
8. Let T be the value of t when $y = 0$. Then because we see that both (77) and (75) involve $(32 + 100k)e^{-kt}$ we try to see what results from taking the value of this quantity from (77) and substituting it in (75). From (77) with $y = 0$ we find that $(32 + 1000k)e^{-kt} = -32kT + 32 + 1000k$. If we substitute this value in (75) we find that $v = 1000 - 32T$. This is the value of v when the object reaches the ground. In the case of no air resistance v is also $1000 - 32T$ but in this case, the value of T is $1000/16$ and so $v = -1000$.

9.(a) Take the upward direction as positive. During the upward motion the resistance is downward and so is $-Kv^2$ where K is positive. Then by Newton's second law $m\dot{v} = -32m - Kv^2$.

(b) No. During the downward motion the air resistance is upward and should be positive. But $-Kv^2$ is negative.

10.(a) Given $\dot{v} = -32 - kv^2$, $dt/dv = -1/(32+kv^2)$. This form suggests an arc tan integral (Ch. 11, sec). Write $t = -(1/32)\int dv/[1+(k/32)v^2]$ and let $u = \sqrt{k/32}\,v$. Since $du/dv = \sqrt{k/32}$ we write $t = (-1/32)\sqrt{32/k}\int 1/(1+u^2)\sqrt{k/32}\,dv = -1/\sqrt{32k}\int du/[1+u^2]$. Then $t = -1/\sqrt{32k}\tan^{-1}u + C$. Now replace u by its value and solve for v.

(b) Using the result of (a) let $v = 1000$ when $t = 0$. Then $v = \sqrt{32/k}\tan(\alpha - \sqrt{32k}\,t)$ where $\alpha = \tan^{-1}\sqrt{k/32}\,1000$.

(c) $\alpha = \tan^{-1}\sqrt{(0.005/32)}\,1000 = \tan^{-1}12.5$. Hence $\alpha = 85°25' = 1.491$ rad. approx. Then $v = 80\tan(1.491 - 0.4t)$. Now $v = 0$ when $t = 3.7$ sec. approx.

11.(a) We have $\dot{y} = \sqrt{32/k}\tan(\alpha - \sqrt{32k}\,t)$. Replace tan by sin/cos and let $u = \cos(\alpha - \sqrt{32k}\,t)$. Then $du/dt = \sqrt{32k}\cdot\sin(\alpha - \sqrt{32k}\,t)$. Hence write $y = \int (1/k)[\sin(\alpha - \sqrt{32k}\,t)/\cos(\alpha - \sqrt{32k}\,t)]\sqrt{32k}\,dt = (1/k)\int (1/u)(du/dt)dt = (1/k)\int (1/u)du = (1/k)\log u + C = (1/k)\log\cos(\alpha - \sqrt{32k}\,t) + C$.

(b) If we measure height from the ground up then $y = 0$ when $t = 0$. Then $C = -(1/k)\log\cos\alpha$.

(c) We wish to calculate $y = (1/k)\log\cos(\alpha - \sqrt{32k}\,t) - (1/k)\log\cos\alpha$ when $\alpha = 1.491$ rad. (see exercise 10(c)) and $k = 0.005$. Now $\log\cos 1.491 = 8.90260 - 10 = -1.0974$. Hence $y = 200\log\cos(1.491 - 0.4t) - 200(-1.0974)$. We know from exercise 10(c) that the time to reach max. ht. is 3.7 sec. Hence max. ht. = 219.5 ft.

12. From exercise 10(b) we know the formula for the velocity at any time t. At maximum height $v = 0$. Hence $\alpha - \sqrt{32k}\,t = 0$ where $\alpha = \tan^{-1}(\sqrt{k/32}\,1000)$. Hence $t_1 = (1/\sqrt{32k})\alpha$.

13. From (11a) and (11b) we have the formula for the height at any time t. In exercise 12 we calculated the time to reach maximum height when $v_0 = 1000$. Using the value $t_1 = (1/\sqrt{32k})\tan^{-1}\alpha$, substitute this in the formula for height. This gives $y_1 = (1/k)\log\cos 0 - (1/k)\log\cos\alpha = (1/k)\log\cos[\tan^{-1}\sqrt{k/32}\,1000]$. The cosine of the angle whose tangent is $\sqrt{k/32}\,1000$ is $1/\sqrt{1+(k/32)1000^2}$. Then $y_1 = -1/k\,\log(1/\sqrt{1+k/32)1000^2}) = (1/2k)\log[1+(k/32)1000^2]$.

CHAPTER 12, SECTION 8, FIRST SET

1. (a) Since $\tanh x = \sinh x/\cosh x$, if we substitute $-x$ for x in (78) and (79) we see that $\sinh(-x) = -\sinh x$ and $\cosh(-x) = \cosh x$.

(b) Replace $\tanh(x/2)$ by $\sinh(x/2)/\cosh(x/2)$ in numerator and denominator and simplify. This gives $\sinh x = 2\sinh(x/2)\cosh(x/2)$. Now use the definitions of $\sinh x$ and $\cosh x$ in (78) and (79) to form the right side. Multiplication gives $\sinh x$.

(c) Replace $\tanh x$ by $\sinh x/\cosh x$ and simplify. Use $\cosh^2 x - \sinh^2 x = 1$.

(d) Use the definitions $\sinh x$ and $\cosh x$ in (78) and (79) and form $2\sinh x\cosh x$.

(e) Use the definitions of $\sinh x$ and $\cosh x$ in (78) and (79) to form $\cosh^2 x + \sinh^2 x$.
(f) Use (e) and replace $\sinh^2 x$ by $\cosh^2 x - 1$.
(g) Use (e) and replace $\cosh^2 x$ by $1 + \sinh^2 x$.
(h) $\tanh 2x = \sinh 2x/\cosh 2x = 2 \sinh x \cosh x/(\cosh^2 x + \sinh^2 x)$ by (d) and (e). Now divide numerator and denominator by $\cosh^2 x$.
(i) Replace each function by the definitions in (78) and (79). Of course $\sinh(x + y) = [e^{x+y} - e^{-(x+y)}]/2$.
(j) Proceed as in (i).

2. (a) Since $\sinh x = (e^x - e^{-x})/2$, if we differentiate with respect to x we obtain $(e^x + e^{-x})/2$ which is $\cosh x$.
(b) Write $\tanh x = \sinh x/\cosh x$ and differentiate the quotient. Use the result of (a) and (80) to carry out the differentiations.
(c) Write $\coth x = \cosh x/\sinh x$ and differentiate the quotient, using (a) and (80) to carry out the differentiations.

3. (a) Use the definition (96) of $\sinh^{-1} x$ and differentiate.
(b) Use the definition (97) of $\cosh^{-1} x$ and differentiate.
(c) Use the definition (98) of $\tanh^{-1} x$ and differentiate.

CHAPTER 12, SECTION 8, SECOND SET

1. To calculate the velocity we use (111) or (116) with $k = 0.005$ and $t = 3.7$. Then $y = 80[(e^{2.96}-1)/(e^{2.96}+1)] = 72$ ft. approx. To calculate the distance fallen in 3.7 sec. we use (115) or (118) with $k = 0.005$ and $t = 3.7$. Then $y = 200 \log[(e^{1.48}+e^{-1.48})/2] =$ 171 ft. approx.
2. We have from (115) that $219.5 = (1/k)\log[(a+1/a)/2]$ or $219.5k = \log[(a^2 + 1)/2a]$ or $e^{1.1} = (a^2 + 1)/2a$ or $3 = (a^2 + 1)/2a$. If we solve for a we get $a = 0.17$ and $a = 5.83$. Then $e^{\sqrt{32k}t} = 0.17$ and $e^{\sqrt{32k}t} = 5.83$. Taking logs of both sides we see that the first value of t is negative (which has no physical meaning, and the second yields $t = 4.4$ sec. approx.

   On first thought we might argue that the object should fall the 219.5 feet faster than rise 219.5 feet because on the way down gravity helps the motion whereas it hinders the upward motion; the air resistance hinders the motion both ways. However the object is shot up with a velocity of 1000 ft/sec and covers some distance before losing a good deal of its velocity. On the way down the object starts with 0 velocity and air resistance opposes what little velocity the object gains. Even if the object took 3.7 seconds to fall and there were no air resistance it would attain a velocity of only 32(3.7) ft/sec. It cannot cover much distance because then its average velocity would be $\frac{1}{2}(32)(3.7)$ or about 60 ft/sec and in 3.7 seconds it would fall 222 ft. Hence with no air resistance it would cover only slightly more than the 219.5 feet. The retardation due to air resistance of $.005v^2$ is small but this is an acceleration which constantly reduces the velocity and so the object should take more time to fall. We can also argue that the terminal velocity, $\sqrt{32/k}$, amounts to 80 ft/sec in this case. This shows that the object doesn't come near reaching the 1000 ft/sec velocity with which it is shot up.

3. We use (111) to calculate the velocity when $t = 4.38$ sec. and $k = 0.005$. We have $\dot{y} = \sqrt{6400}\,(e^{2(.4)(4.38)} - 1)/(e^{2(.4)(4.38)} + 1) = 80(33.19 - 1)/(33.19 + 1) = 75.2$ ft/sec. We see how little velocity the object acquires by the time it hits the ground compared with the 1000 ft/sec with which it was shot up.
4. We have $v\,dv/dy = 32 - kv^2$ or $dy/dv = v/(32 - kv^2)$, Let $u = 32 - kv^2$. Then $y = (-1/2k)\log(32 - kv^2) + C$. When $y = 0$, $v = 0$. Hence $C = (1/2k)\log 32$. Then $y = (1/2k)\log 32 - (1/2k)\log(32 - kv^2)$, which is the text's answer.
5. We start with the step in Exercise 4 where $y = (-1/2k)\log(32 - kv^2) + C$; only now when $y = 0$, $v = v_0$. Then $C = (1/2k)\log(32 - kv_0^2)$. With this value of C the answer is that in the text.
6. From the answer to Exercise 5 (see text) $e^{2ky} = (32 - kv_0)^2/(32 - kv^2)$. Then solving for v gives $v = \sqrt{(32e^{2ky} + kv_0^2 - 32)/ke^{2ky}}$. Now we see from (115) that as t becomes infinite, y becomes infinite. The terms involving y in the expression for v are $32e^{2ky}$. These become infinite as y does and the radicand approaches $32/k$. Hence the terminal velocity is $\sqrt{32/k}$.
7. We know from Exercise 6 that $\sqrt{32/k} = 16$. Hence $k = 1/8$. Now we use (111) to find t when $k = 1/8$ and $y = 15.8$ sec. Then $15.8 = 16(e^{4t} - 1)/(e^{4t} + 1)$. Solve for e^{4t}. $e^{4t} = 159$. Then $4t = \log 159$. The $\log 159$ is not in our natural log tables but we see from table V that $e^5 = 148.4$ and $e^6 = 403.4$. Hence $\log 159 = 5.04$ approx. Hence $t = 1.26$ sec approx.
8. The terminal velocity is $\sqrt{32/k}$. Hence we have but to calculate this quantity for the two given values of k.
9. We can use the result of Exercise 4 which states that $y = (1/2k)\log[32/(32 - kv^2)]$. The v in one case is $.95(170) = 161.5$ and in the other case it is $.95(15) = 14.25$. Substitute these values of v in the formula and calculate y. The answers are in the text.
10. From Exercise 13 of Section 7, with V replacing 1000, we have $y_1 = (1/2k)\log[1 + (V^2k/32)]$. This is the height to which the object will rise. Now to calculate the velocity it acquires in falling this distance we use the result of Exercise 4 in this list and first solve for v in terms of y. We obtain $v^2 = 32/k - (32/k)e^{-2ky}$. If we now substitute y_1 for y we have $v^2 = 32/k - (32/k)[32/(32 + V^2k)]$. We have used the fact that $e^{-\log z} = e^{\log z^{-1}} = z^{-1} = 1/z$. Hence $v^2 = 32V^2/(32 + V^2k)$, which equals the result in the text.

CHAPTER 12, SECTION 9

(a) Write $\log y = x \log x^2 = 2x \log x$. Then $(1/y)y' = (2x)(1/x) + 2 \log x = 2 + 2 \log x$. Then $y' = (2 + 2 \log x)(x^2)^x$.

(b) Write $\log y = 2x \log x^2 = 4x \log x$. Then $(1/y)y' = 4 + 4 \log x$. $y' = (4 + 4 \log x)(x^2)^{2x}$.

(c) Write $\log y = \log \sqrt{2x + 5} - \log \sqrt{x^2 + 7} = \frac{1}{2} \log (2x + 5) - \frac{1}{2} \log (x^2 + 7)$. Then $(1/y)y' = [1/(2x + 5)] - [x/(x^2 + 7)]$. Hence $y' = [1/(2x+5) - x/(x^2+7)][\sqrt{2x + 5}/\sqrt{x^2 + 7}]$.

(d) $\log y = 3 \log(x^2+2)+4 \log(1-x^3)$. Then $(1/y)dy/dx = 6x/(x+2)+12x^2/(1-x^3)$. Then $dy/dx = (x^2+2)^3(1-x^3)^4 [6x/(x+2)+12x^2/(1-x^3)]$.

(e) Write $\log y = 6 \log(x^2+1)+4 \log(3x^3-5)+3 \log(x^4-2x)$. Then $(1/y)y' = 12x/x^2+1)+36x^2/(3x^2-5)+(12x^3-6)/x^4-2x)$. Multiplication of both sides by y gives the result.

(f) $\log y = \log x+2 \log(1-x^2)-1/2 \log(1+x^2)$. Then $(1/y)y' = 1/x-4x/(1-x^2)-x/(1+x^2)$. One now multiplies by y as in (d).

Solutions to Chapter 13

CHAPTER 13, SECTION 1

1. In each case we use (8). (b) $dy = \frac{1}{2}x^{-1/2}dx$; (d) $dy = \cos x\,dx$; (f) $dy = (1/x)dx$; (g) $dy = e^x\,dx$; (i) $dy = 4(x^2+1)^3 2x\,dx$.
2. If $y = \sqrt{x}$, $dy = 1/2\sqrt{x}\,dx$. When $x = 100$ and $dx = 1$, $dy = .05$. Also, at $x = 100$, $y = 10$. dy is merely the (approximate) increase in y when x changes from 100 to 101. Hence the answer is 10.05.
3. If $y = \sin x$, $dy = \cos x\,dx$. Let $x = 60°$ and $dx = -1° = -0.01745$ rad. Then $dy = \frac{1}{2}(-0.01745) = -.0087$. At $x = 60°$, $\sin x = \sqrt{3}/2 = 0.866$. Hence $y + dy = 0.8573$.
4. Since $V = \frac{4}{3}\pi r^3$, $dV = 4\pi r^2\,dr$. When $r = 3$ and $dr = 0.2$, $dV = 7.2\pi$.
5. The width $x = 50\cot A$ where A is the angle of elevation. Then $dx = -50\csc^2 A\,dA$. When $A = 45°$ and $dA = 20' = 0.00582$ rad, $dx = -50\cdot 2(0.00582) = 0.582$, which is about $^{100}/_{173}$.
6. $dT = (2\pi/\sqrt{32})(1/2\sqrt{\ell})\,d\ell$. We are given that $d\ell/\ell = 0.01$.
   (a) The approximate error in the period is $(\pi/\sqrt{32})(\sqrt{\ell})(d\ell/\ell)$ and since $d\ell/\ell = 0.01$, $dT = \pi\sqrt{\ell/32}\,(.01) = 2\pi\sqrt{\ell/32}\,(.005) = .005T$. Hence $dT/T = .005$.
   (b) dT/T is the error in T, and T is the number of seconds in a period. Hence dT/T is also the error per second.
   (c) The error in one day is $60\cdot 60\cdot 24(.005) = 432$ seconds.
7. When $r = s\cos A$, $dr = -s\sin A\,dA$ and when $r = h\cot A$, $dr = -h\csc^2 A\,dA$. Since $\sin A$ is small when A is small, whereas $\csc A$ is very large, the first value of dr would be smaller.
8. When $s = h\csc A$, $ds = -h\csc A\cot A\,dA$ and when $s = r\sec A$, $ds = r\sec A\tan A$. When A is small, $\sec A\tan A$ is much smaller than $\csc A\cot A$. Hence the latter formula, $s = r\sec A$, is better.
9. Since $T = 2\pi\sqrt{\ell/g}$ and g is the independent variable, $dT = 2\pi\sqrt{\ell}\,(-1/2g^{3/2})\,dg = (-T/2g)\,dg$. In one day $T = 86{,}400$ sec. Then, since $dT = 20$, $dg = -(2\,dT/T)g = g/2160$.
10. The arithmetic in this problem is easier if we recognize first that we want the error per second; this is dT/T. Hence we calculate this and find $dT/T = (1/2\ell)\,d\ell$. Since $\ell = 3$ and $d\ell = \frac{1}{96}$ (in feet), $dT/T = \frac{1}{6}\cdot 96$. This is the error per second. Multiply by $3600\cdot 24$ to get the error in seconds for one day or by $60\cdot 24$ to get the error in minutes.
    Ans. $2\frac{1}{2}$ minutes.
11. (a) If g is the independent variable, $dT = (-T/2g)\,dg$. See Exercise 9. Then $dT/T = -dg/2g$. If $dg/g = 0.002$, then $dT/T = 0.001$.
    (b) We start with $T = 2\pi\sqrt{\ell/g}$. Then $g = 4\pi^2\ell/T^2$. Since T is constant, $dg = (4\pi^2/T^2)\,d\ell = g(d\ell/\ell)$. Now $d\ell/\ell = 0.005$. Then $dg/g = 0.005$.

(c) From (b) we have $g = 4\pi^2\ell/T^2$. Now T is the independent variable. Then $dg = -8\pi^2\ell T^{-3}dT = -2(4\pi^2\ell/T^2)(1/T)\,dT = -2g(dT/T)$. We are given that $dT/T = 0.001$. Then $dg/g = -0.002$.

12. $V = \pi r^2 h$. Then $h = V/\pi r^2$. Hence $dh = (-2V/\pi r^3)\,dr$. We have that $r = 5$ and $h = 10$ and so $V = 250\pi$. Also $dr = 0.05$. Hence $dh = 0.2$ in.
13. $dy = (\sec^2 x/\tan x)\,dx$. When $x = 60°$ and $dx = 1' = 0.000291$ rad, $dy = (4/\sqrt{3})(0.000291) = 0.0007$.
14. $dR = (-V^2\sin 2A/g^2)\,dg = -(V^2\sin 2A/g)(dg/g) = -R\,dg/g$. Hence $dR/R = -dg/g$.
15. dy/dx means $\lim_{x\to 0} \Delta y/\Delta x$ or $f'(x)$ or y'. It also means the quotient of dy by dx where $dy = f'(x)\,dx$.

CHAPTER 13, SECTION 2

1. We have $f(b) - f(a) = f'(\bar{x})(b - a)$ where $a < \bar{x} < b$. Since $f'(\bar{x}) > 0$, $f(b) > f(a)$.
2. We wish to find the $\bar{x}$ for which $4^2 - 1^2 = 2\bar{x}(4 - 1)$. Then $\bar{x} = 2\tfrac{1}{2}$.
3. Here we have $4^3 - 1^3 = 3\bar{x}^2(4 - 1)$. Hence $\bar{x} = \sqrt{7}$. The value $-\sqrt{7}$ does not lie in the interval (1, 4).
4. By applying the mean value theorem we have $(1/1) - (1/-1) = (-1/\bar{x}^2)\cdot(1 - (-1))$ or $\bar{x}^2 = -1$. Hence there is no $\bar{x}$ between -1 and $+1$. The theorem fails because the function is not continuous in the interval $(-1, 1)$.
5. By the mean value theorem, $\sin x - \sin 0 = \cos\bar{x}(x - 0)$. Now $\cos\bar{x} < 1$ for $x > 0$ and $< \pi/2$. Hence $x > \sin x$ for $0 < x < \pi/2$. For $x \geq \pi/2$, x is necessarily $> \sin x$ because $\sin x \leq 1$.
6. If $y = \log x, y' = 1/x$. Then $\log(1 + h) - \log 1 = [1/(1 + \bar{h})](1 + h - 1)$. Since $1 + \bar{h}$ lies between 1 and $1 + h$, $\bar{h} < h$. Hence $\log(1 + h) > h/(1 + h)$. Likewise $1/(1 + \bar{h}) < 1$. Hence $\log(1 + h) < h$.
7. $e^x - e^0 = e^{\bar{x}}(x - 0)$. Now $\bar{x} > 0$; hence $e^{\bar{x}} > 1$. Then $e^x - e^0 > x$ and $e^x > 1 + x$.
8. $ab^2 + bb + c - (aa^2 + ba + c) = (2a\bar{x} + b)(b - a)$. Then $a(b^2 - a^2) + b(b - a) = (2a\bar{x} + b)(b - a)$. Divide through by $(b - a)$. Then $ab + a^2 + b = 2a\bar{x} + b$. Hence $\bar{x} = (b + a)/2$.
9. The average velocity.

CHAPTER 13, SECTION 3, FIRST SET

1. (a) Apply (22) directly.
   (b) Apply (22) directly.
   (c) Differentiating the numerator gives $(1/x/n)(1/n) = 1/x$. Differentiating the denominator gives -1. Hence $f'(x)/g'(x) = -1/x$ and as x approaches n the limit is $-1/n$.

(d) Apply (22) directly. Ans. 0.
(e) Applying (22) once gives another 0/0 form. Hence apply (22) again to the new form. Ans. $\frac{1}{2}$.
(f) Apply (22) directly. Ans. 1.
(g) Applying (22) gives $(\sec^2 x - 1)/(1 - \cos x)$. This is indeterminate at $x = 0$. Apply (22) again. This gives $2 \sec x \sec x \tan x/\sin x$. Now $\tan x = \sin x/\cos x$. Hence the expression becomes $2 \sec^2 x/\cos x$ and as $x \to 0$ this approaches 2.
(h) The given expression is the derivative of $\sin x$ at $x = a$. Hence the answer is $\cos a$.
(i) Apply (22) directly.
(j) Apply (22) directly.

2. The original function is not an indeterminate form. Hence L'Hospital's rule does not apply.
3. L'Hospital's rule calls for the existence of $\lim [f'(x)/g'(x)]$. If this limit does not exist, and it doesn't in the problem in question, we cannot conclude anything by the rule. We may however find the limit of the original function by some other means and we can.
4. As k approaches 0 in the expression for v we approach a 0/0 form. Differentiate numerator and denominator with respect to k. This gives $(32t\,e^{-kt})/1$. As k approaches 0 the expression approaches 32t.
5. Differentiate numerator and denominator with respect to k. This gives $32(t - t\,e^{-kt})/2k$. As k approaches 0 the expression is indeterminate. Hence apply (22) again and we obtain as the limit $16t^2$.
6. Apply (22) by differentiating with respect to k. This gives $100t/(1 + 100kt)$. As k approaches 0 we get 100t which is the correct formula for distance traveled in a vacuum.
7. Write the given expression as $(32 - 32e^{-kt} + V e^{-kt}k)/k$. Then as k approaches 0 the expression approaches a 0/0 form. Apply (22). This gives $+32t\,e^{-kt} + V e^{-kt} - kt\,V e^{-kt}$. As k approaches 0 the expression approaches $32t + V$.

CHAPTER 13, SECTION3 , SECOND SET

1. (a) Write the expression as $\tan 2x/\tan x$ and apply (22).
   (b) Apply (33). This gives $\lim\limits_{x \to \infty} 5x^4/e^x$ and is still indeterminate. But we can see that repeated application of (33) will give $5!/e^x$ and the limit is 0.
   (c) This is a 0/0 form. Apply (22). Ans. $\frac{1}{3}$.
   (d) Compare (b). Apply (33) n times. Ans. 0.
   (e) Applying (33) gives $\sec x \tan x/\sec^2 x = \tan x/\sec x = \sin x$. Hence the limit is 1.
   (f) Applying (33) gives $\cos x/\sin x \big/ \sec^2 x/\tan x = 1/\sec^2 x$. As x approaches 0 the limit is 1.

(g) Differentiating numerator and denominator gives $-2/(1-2x)\big/\pi\sec^2\pi x$. Write this as $-2\cos^2\pi x/\pi(1-2x)$. For x approaching $\frac{1}{2}$, this is a $0/0$ form. Hence apply (22).

(h) We must apply (33) n times. Then only a_0/b_0 will remain.

(i) If we apply (33) m times, the numerator will be a constant and the denominator will still contain positive integral powers of x. Hence the limit will be 0.

2. Since the original function and its reciprocal have the same limit, $\ell = 1/\ell$. Hence $\ell = \pm 1$. Since the original function is positive, $\ell = 1$.

CHAPTER 13, SECTION 3, THIRD SET

1. (a) Write the given expression as $\log x/1\big/\sqrt{x}$, differentiate numerator and denominator and simplify.

(b) Write as $t/e^{1/t}$. The numerator approaches 0 and the denominator becomes infinite. Hence the limit is 0.

(c) Write the given expression as $(x^2-\sin^2 x)/x^2\sin^2 x$. This is a $0/0$ form. Apply (22) four times. The differentiation is straightforward though one can save algebra by merely adding terms at several stages. The answer is $\frac{1}{3}$.

(d) Write as $\tan(3/x)\big/1/x$. This is a $0/0$ form. Apply (22) once. This yields $3\sec^2(3/x)$ which has the limit 3 as x approaches ∞.

(e) Write as $(t\log t - t + 1)/(t-1)\log t$. This is a $0/0$ form. Apply (22) twice and we obtain the limit $\frac{1}{2}$.

(f) This form is not indeterminate. $e^{-1/t}$ approaches 1 and the factor t becomes infinite.

(g) This is a $0\cdot\infty$ form. Write it as $\log\tan x\big/1/x$. Apply (22). This gives $-x(x/\tan x)\sec^2 x$. Now $x/\tan x$ approaches 1 [It is $\cos x(x/\sin x)$.] Hence the entire expression approaches 0.

(h) This is an $\infty-\infty$ form. Write it as $(\sin 3x\cos x - \cos 3x\sin x)/\cos x\cos 3x$. Now it is a $0/0$ form. But $\sin 3x\cos x - \cos 3x\sin x = \sin(3x-x)$. Hence we have $\sin 2x/\cos x\cos 3x$. But $\sin 2x = 2\sin x\cos x$. Hence we have $2\sin x/\cos 3x$ and as x approaches $\pi/2$ the expression becomes infinite.

(i) Write $\log y = x\log\sin x$. Now write $x\log\sin x$ as $\log\sin x\big/1/x$. Apply (22). This gives $\cot x\big/-1/x^2 = -x^2/\tan x$. Apply (22) again and the limit is 0. Since $\log y$ approaches 0, y approaches 1.

(j) Let $y = x^{1/x}$. Then $\log y = (1/x)\log x$. Now $1/x$ becomes infinite and $\log x$ becomes infinite. Hence $\log y$ and so y becomes infinite.

(k) This is a 1^∞ form. Write $\log y = \cot x\log(3\sin x + \cos x)$. The right side is $\infty\cdot 0$ form. Write it as $\log(3\sin x+\cos x)/\tan x$. This is a $0/0$ form. Apply (22). This limit is 3. Hence $y = e^3$.

(l) This is a 1^∞ form. Write $\log y = (1/x)\log(e^x + 2x)$. The right side is an $\infty \cdot 0$ form. Write it as $\log(e^x + 2x)/x$. Now it is a 0/0 form. Apply (22). The limit is 3. Hence $y = e^3$.

(m) This is an ∞^0 form. Write $\log y = \cos x \log \sec x$. The right side is a $0 \cdot \infty$ form. Write it as $\log \sec x/\sec x$. Now it is an ∞/∞ form. Apply (33). Then the limit is 0 and $y = 1$.

(n) This is a 1^∞ form. Write $\log y = [1/(1 - x)]\log x$. Now it is an $\infty \cdot 0$ form. Write the right side as $\log x/(1 - x)$ and we have a 0/0 form. Apply (22). Then the limit is -1 and $y = e^{-1}$.

(o) This is an ∞^∞ form. It is not indeterminate because if the base and exponent both grow larger the entire power grows larger. The answer is ∞.

(p) Write $\log y = [1/(x - 1)]\log(x - 1)$. Both factors becomes infinite as $x - 1$. Hence $\log y$ and y become infinite.

(q) This is an $\infty - \infty$ form. Write it as $\log[(1 + x)/x] = \log[(1/x) + 1]$. As x becomes infinite the number $1/x + 1$ approaches 1 and the logarithm approaches 0.

(r) This is a $0 \cdot \infty$ form. Write it as $\log 2x/\csc x$. This is an ∞/∞ form. Apply (33) and this gives $1/x/-\csc x \cot x$. Write this as $-\sin x \tan x/x$. This is a 0/0 form. Apply (22). This gives $-\sin x \sec^2 x - \cos x \tan x$. The limit is 0.

(s) This is an $\infty - \infty$ form. Write it as $(x - \sin x)/x \sin x$. Now it is a 0/0 form. Apply (22) twice. The limit is 0.

(t) This is an ∞^0 form. Write $\log y = (1/x)\log(1 + x)$. The right side is a $0 \cdot \infty$ form. Write it as $\log(1 + x)/x$. This is an ∞/∞ form. Apply (33). The limit is 0. Hence $y = 1$.

(u) This is a $0 \cdot \infty$ form. Write it as $x/\tan x$. We know this approaches 1.

(v) This is an $\infty - \infty$ form. Write it as $(1 - \sin x)/\cos x$ and now it is a 0/0 form. Apply (22) and the limit is 0.

(w) This is a 0^0 form. Then $\log y = (a/\log x)\log x = a$. Then $y = e^a$.

2. Yes. If the base is getting larger and larger and so is the exponent, the entire power becomes infinite.
3. Yes. We have a form $[f(x)]^{g(x)}$ where $f(x)$ approaches 0 and $g(x)$ becomes infinite as, say, x approaches a. Then $\log y = g(x)\log(f(x)$. Now $\log f(x)$ becomes infinite and $g(x)$ is also becoming infinite. Then $\log y$ becomes infinite and so does y. Hence a 0^∞ form yields ∞ always. See Exercises 1(j) and 1(p).
4. Following the suggestion we add and obtain $v = (-32 + 32e^{-kt} + 1000ke^{-kt})/k$. This is now a 0/0 form. Apply (22) with k as the variable. This gives $-32te^{-kt} + 1000e^{-kt} - 1000kt\,e^{-kt}$. As k approaches 0, v approaches $-32t + 1000$.
5. As the expression stands it is a combination of indeterminacies. If we add the fractions we get $y = (-32kt + 32 - 32e^{-kt} + 1000k - 1000ke^{-kt})/k^2$. For $k = 0$, this is a 0/0 form. Hence apply (22) twice. The limit as k approaches 0 is $-16t^2 + 1000t$.

6. Using the similar triangles BNP and BAM we have AB/MA = NB/NP. Introduce the angle θ = angle NOP. Then AB = r + OB, MA = rθ, NB = NO + OB = r cos θ + OB, NP = r sin θ. If we substitute these values in the equation and solve for OB we have OB = $(\theta r \cos\theta - r\sin\theta)/(\sin\theta - \theta)$. This is a 0/0 form. We apply (22). Simplify. Then apply (22) again. Then lim OB = 2r.

Solutions to Chapter 14

CHAPTER 14, SECTION 2

1. (a) Let $u = x$, $v' = \cos x$. Then $u' = 1$, $v = \sin x$. Hence
$\int x \cos x\, dx = x \sin x - \int \sin x\, dx = x \sin x + \cos x + C$.

(b) Let $u = \log x$, $v' = x$. Then $u' = 1/x$, $v = x^2/2$. Hence
$\int x \log x\, dx = (x^2/2) \log x - \int (x^2/2)(1/x)\, dx = (x^2/2) \log x - (x^2/4) + C$.

(c) Let $u = x$, $v' = e^x$. Then $u' = 1$, $v = e^x$. Hence
$\int xe^x\, dx = xe^x - \int e^x\, dx = xe^x - e^x + C$.

(d) Let $u = x^2$, $v' = e^x$. Then $u' = 2x$ and $v = e^x$. Hence
$\int x^2e^x\, dx = x^2e^x - \int e^x\, 2x\, dx$. This latter integral must again be evaluated by integration by parts. The result of this is twice the answer in (c). Hence the complete answer is $x^2e^x - 2xe^x + 2e^x + C$.

(e) Let $u = e^x$, $v' = \sin x$. Then
$\int e^x \sin x\, dx = -\cos x e^x + \int \cos x\, e^x\, dx$. To evaluate the latter integral let $u = e^x$, $v' = \cos x$. Then $\int \cos x\, e^x\, dx = e^x \sin x - \int \sin x e^x\, dx$. Substitute this result in the first one and transpose $\int \sin x e^x\, dx$ to the left side. Divide by 2.

(f) Let $u = \log x$, $v' = 1$. Then
$\int \log x\, dx = x \log x - \int x(1/x)\, dx = x \log x - x + C$.

(g) Let $u = x$, $v' = e^{ax}$. Then $u' = 1$, $v = (1/a)e^{ax}$. Hence
$\int xe^{ax}\, dx = (x/a)e^{ax} - \int (1/a)e^{ax}\, dx = (xe^{ax}/a) - (e^{ax}/a^2) + C$.

(h) Let $u = e^{ax}$, $v' = \sin nx$. Then $u' = ae^{ax}$, $v = (-1/n) \cos nx$. Hence
$\int e^{ax} \sin nx\, dx = -(1/n)e^{ax} \cos nx + \int (a/n) \cos (nx)e^{ax}\, dx$. To do the latter integration let $u = e^{ax}$, $v' = \cos nx$. Then $\int \cos (nx)e^{ax}\, dx$
$= e^{ax}(1/n) \sin nx - \int (a/n) \sin (nx)e^{ax}\, dx$. If we substitute this latter result in the first one and transpose the integral we get on the left side
$[(n^2 + a^2)/n^2] \int e^{ax} \sin x\, dx$. Divide through by $(n^2 + a^2)/n^2$ and we get
$\int e^{ax} \sin nx\, dx = [e^{ax}/(a^2 + n^2)](a \sin nx - n \cos nx) + C$.

(i) Let $u = x$, $v' = \sin 2x$ and apply the formula.
(j) Let $u = x$, $v' = \sec^2 2x$. Then

$\int x \sec^2 2x\,dx = (x \tan 2x)/2 - \frac{1}{2}\int \tan 2x\,dx$. But $\int \tan 2x\,dx$

$= \int (\sin 2x/\cos 2x)\,dx = -\frac{1}{2}\log|\cos 2x|$. Hence the answer is

$[(x \tan 2x)/2] + \frac{1}{4}\log|\cos 2x| + C$.

(k) Let $u = x$, $v' = \sin^2(x/2)$. Then $u' = 1$ and $v = \frac{1}{2}(x - \sin x)$. Then

$\int x \sin^2(x/2)\,dx = x^2/2 - (x - \sin x)/2 - \int [(x/2) - (\sin x)/2]\,dx$

$= x^2/4 - [(x \sin x)/2] - (\cos x)/2 + C$.

(l) Let $u = \sin x$, $v' = \cos 3x$. Then

$\int \sin x \cos 3x\,dx = \sin x \sin 3x/3 - \int [(\sin 3x \cos x)/3]\,dx$. Now evaluate the latter integral by parts. Let $u = \cos x$ and $v' = \sin 3x$. Then

$\int [(\sin 3x \cos x)/3]\,dx = -(-\cos x \cos 3x)/9 - \int [(\cos 3x \sin x)/9]\,dx$. If we substitute this result in the first integration and transpose the integral we get $[(3 \sin x \sin 3x)/8] + [(\cos x \cos 3x)/8] + C$.

(m) Let $u = \arcsin x$, $v' = 1$. Then $u' = 1/\sqrt{1 - x^2}$, $v = x$. Then

$\int \arcsin x\,dx = x \arcsin x - \int (x/\sqrt{1 - x^2})\,dx = x \arcsin x + \sqrt{1 - x^2} + C$.

(n) Let $u = \arctan x$, $v' = 1$ and proceed as in (m). The answer is $x \arctan x - \frac{1}{2}\log(1 + x^2) + C$.

(o) Let $u = x e^x$, $v' = 1/(1 + x)^2$. $u' = e^x + xe^x$, $v = -1/(1 + x)$. Then

$\int [(x e^x)/(1 + x)^2]\,dx = -(x e^x)/(1 + x) + \int [(e^x - x e^x)/(1 + x)]\,dx$

$= -(x e^x)/(1 + x) + \int [(e^x)(1 + x)]/(1 + x)\,dx = -(x e^x)/(1 + x) + e^x + C$

$= [e^x/(1 + x)] + C$.

(p) Follow the procedure of (k).

(q) Let $u = x^2$, $v' = \cos x$. Then $\int x^2 \cos x\,dx = x^2 \sin x - \int 2x \sin x\,dx$. To integrate the latter integral let $u = 2x$, $v' = \sin x$. This gives the text's answer.

2. The constant of integration has been omitted. The right side of the integration by parts should read $1 + \int dx/x + C$.

3. (a) Same reason as in Exercise 2.
   (b) Write $u = \log x$. Then the given integrand is in the form $(1/u)\,du/dx$. Hence the text answer.

4. From $\sec x = \cosh u$ we have by implicit differentiation with respect to u, $\sec x \tan x(dx/du) = \sinh u$. Then in differential form $\sec x\,dx$

$= (\sinh u)\,du/\tan x$. We must convert $\tan x$ to an expression in u. But $\tan^2 x = \sec^2 x - 1 = \cosh^2 u - 1 = \sinh^2 u$. Hence $\tan x = \sinh u$. Then

$\sec x\,dx = du$ and $\int \sec x\,dx = \int du = u + C = \cosh^{-1}(\sec x) + C$. Since $\sinh^2 u = \cosh^2 u - 1 = \sec^2 x - 1 = \tan^2 x$, $\sinh u = \tan x$ and $u = \sinh^{-1}(\tan x)$. Hence this is also a form of the integral $u + C$.

5. Follow the method of Exercise 4 with appropriate changes since here we have $\csc x$ in place of $\sec x$.
6. Let $y = \log(\sec x + \tan x) = \log(\sec x + \sqrt{\sec^2 x - 1})$. Then $e^y = \sec x + \sqrt{\sec^2 x - 1}$ or $(e^y - \sec x)^2 = \sec^2 x - 1$. Then $\sec x = (e^{2y} + 1)/2e^y = (e^y + e^{-y})/2 = \cosh y$. Hence $\cosh y = \sec x$ and $y = \cosh^{-1}(\sec x)$.

CHAPTER 14, SECTION 3

1. Let $u = \cos^{n-1} x$ and $v' = \sin^m x \cos x$. Then $u' = -(n-1)\cos^{n-2} x \sin x$, $v = (\sin^{m+1} x)/m + 1$. Then $\int \sin^m x \cos^n x\,dx = \dfrac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \int \dfrac{\sin^{m+2} x (n-1)\cos^{n-2} x}{m+1}$. Now replace $\sin^{m+2} x$ by $\sin^m x(1 - \cos^2 x)$ and break up the last integral into a difference of two integrals and transpose the second of these.
2. (a) Let $u = \cos^{n-1} x$, $v' = \cos x$. Then straightforward integration by parts gives the result.
   (b) Replace $\sin^2 x$ in the right-hand integral in (a) by $1 - \cos^2 x$ and transpose the term, $-(n-1)\int \cos^n x\,dx$, which results from the substitution.
3. Repeat (a) and (b) of Exercise 2. Thus to do (a) let $u = \sin^{n-1} x$, $v' = \sin x$ and integrate by parts. Then replace $\cos^2 x$ by $1 - \sin^2 x$ in the right hand integral and transpose the integral containing $\sin^n x$.
4. (a) Use the statement in Exercise 3 to reduce $\int \sin^6 x\,dx$ to an expression involving $\int \sin^4 x\,dx$ and then apply the statement again to replace $\int \sin^4 x\,dx$ by one involving $\int \sin^2 x\,dx$. One can repeat the process once more and replace $\sin^0 x$ by 1 or evaluate $\int \sin^2 x\,dx$ by using $\sin^2 x = (1 - \cos 2x)/2$.
   (b) Use the result in Exercise 2(b) and as in 4(a) apply the formula repeatedly. The answer is $\frac{1}{6}\cos^5 x \sin x + \frac{5}{24}\cos^3 x \sin x + \frac{5}{16}\cos x \sin x + \frac{5}{16}x + C$.
   (c) Use (13) to reduce to $\int \sin^4 x\,dx$ and then apply the result of Exercise 3. Or replace $\cos^2 x$ by $1 - \sin^2 x$ and evaluate both resulting integrals by using Exercise 3.
   (d) Use (13) twice or replace $\cos^4 x$ by $(1 - \sin^2 x)^2$ and use Exercise 3. Ans. $-\frac{1}{8}\sin^7 x \cos x + \frac{3}{16}\sin^5 x \cos x - \frac{1}{64}\sin^3 x \cos x - \frac{3}{125}\sin x \cos x + \frac{3}{128}x + C$.

(e) Use (16) to reduce to an integral involving $\sin^4 x$ and then use Exercise 3.

(f) Here (16) seems to be applicable but fails because $p + 1 = 0$. Replace $\sin^2 x$ by $1 - \cos^2 x$ and multiply out. Then replace $\cos^3 x$ by $\cos x(1 - \sin^2 x)$. Ans. $\log(\sec x + \tan x) - \sin x - \frac{1}{3}\sin^3 x + C$.

(g) Use the result of Exercise 3 in reverse. That is, express $\int \sin^{n-2} x\,dx$ in terms of $\int \sin^n x\,dx$ and then apply this new formula to $\int \sin^{-4} x\,dx$. Ans. $(-\cos x/3\sin^3 x) - (2\cos x/3\sin x) + C$.

CHAPTER 14, SECTION 4, FIRST SET

1\. (a) To obtain the partial fractions set $[A/(x + 2)] + [B/(x - 3)] = 1/(x + 2)(x - 3)$. Clearing of fractions gives $A(x - 3) + B(x + 2) = 1$. When $x = 3$, $5B = 1$; $B = \frac{1}{5}$. When $x = -2$, $-5A = 1$, $A = -\frac{1}{5}$. Hence

$$\int dx/(x - 3)(x + 2) = -\tfrac{1}{5}\int dx/(x + 2) + \tfrac{1}{5}\int dx/(x - 3)$$
$$= -\tfrac{1}{5}\log(x + 2) + \tfrac{1}{5}\log(x - 3) + C.$$

(b) Set $1/x(x - 3)(x + 2) = A/x + B/(x - 3) + C/(x + 2)$. Clear of fractions. Then $A(x - 3)(x + 2) + Bx(x + 2) + Cx(x - 3) = 1$. Let $x = 0$. Then $-6A = 1$ and $A = -\frac{1}{6}$. Let $x = 3$, then $15B = 1$ and $B = \frac{1}{15}$. Let $x = -2$; then $10C = 1$ or $C = \frac{1}{10}$. Hence

$$\int dx/x(x + 3)(x - 2) = -\tfrac{1}{6}\int dx/x + \tfrac{1}{15}\int dx/(x - 3) + \tfrac{1}{10}\int dx/(x + 2)$$
$$= -\tfrac{1}{6}\log x + \tfrac{1}{15}\log(x - 3) + \tfrac{1}{10}\log(x + 2) + C.$$

(c) Write the given integral as $\int dx/(x - 3)(x - 2)$ and follow the method in (a).

(d) Let $(x + 2)/(x - 3)(x - 4) = A/(x - 3) + B/(x - 4)$. Then $x + 2 = A(x - 4) + B(x - 3)$. Let $x = 3$; then $-A = 5$. Let $x = 4$; then $B = 6$. Hence the given integral equals $-5\int dx/(x - 3) + 6\int dx/(x - 4)$ $= -5\log(x - 3) + 6\log(x - 4) + C$.

(e) Let $(x^2 + x + 1)/(x - 1)(x + 2)(x - 3) = A/(x - 1) + B/(x + 2) + C/(x - 3)$. Clear of fractions and then let $x = 1, -2$, and 3 successively. This gives A, B and C and then integrate each term.

(f) $(x^2 + x)/(x^3 + 7x^2 + 6x) = x(x + 1)/x(x + 1)(x + 6) = 1/(x + 6)$. Hence $\int 1/(x + 6)\,dx = \log(x + 6) + C$.

(g) Try $1/\cos x(1 + \cos x) = A/\cos x + B/(1 + \cos x)$. Hence $1 = A(1 + \cos x) + B\cos x$. Let $x = \pi/2$. Then $A = 1$. Let $x = 0$. Then $1 = 2A + B$. Since

$A = 1$, $B = -1$. Then $\int dx/\cos x(1 + \cos x) = \int 1/(\cos x)\,dx$
$- \int 1/(1 + \cos x)\,dx = \int \sec x\,dx - \int [(1 - \cos x)/\sin^2 x]\,dx$
$= \int \sec x\,dx - \int 1/\sin^2 x + \int \cos x/\sin^2 x$. In the second integral replace $1/\sin^2 x$ by $\csc^2 x$. In the third, let $u = \sin x$. Then integrating gives $\log(\sec x + \tan x) + \cot x - 1/\sin x + C$. The last two terms equal $(\cos x - 1)/\sin x$, which is $-\tan(x/2)$ because $\tan(x/2)$ $= \sqrt{(1 - \cos x)/2}/\sqrt{(1 + \cos x)/2} = \sqrt{(1 - \cos x)/(1 + \cos x)}$. Multiply numerator and denominator by $1 - \cos x$.

2. Write $dt/dP = 1/kP(L - P)$. Let $1/kP(L - P) = A/P + B/(L - P)$. Clear of fractions. Then $1 = kA(L - P) + kBP$. Let $P = L$. Then $B = 1/kL$. Let $P = 0$. Then $A = 1/kL$. Hence $t = (1/kL)\int dP/P + (1/kL)\int dP/(L - P)$ or $t + C = (1/kL)\log P - (1/kL)\log(L - P) = (1/kL)\log[P/(L - P)]$. Then $P/(L - P) = De^{kLt}$. Let $P = P_0$ when $t = 0$. Then $D = P_0/(L - P_0)$ and $P/(L - P) = [P_0/(L - P_0)]e^{kLt}$. Solve for P. This gives the text answer.
3. Write $dt/dx = 1/k(a - x)(b - x) = A/(a - x) + B/(b - x)$. Then $1 = kA(b - x) + kB(a - x)$. Let $x = b$. Then $B = 1/k(a - b)$. Let $x = a$. Then $A = 1/k(b - a)$. Then $t = 1/k(b - a)\int dx/(a - x) + 1/k(a - b)\int dx/(b - x)$
$= [(-1)/k(b - a)]\log(a - x) + [1/k(b - a)]\log(b - x)$
$= [1/k(b - a)]\log[(b - x)/(a - x)] + C$. When $t = 0$, $x = 0$. Then $C = -1/k(b - a)\log(b/a)$. Hence $t = [1/k(b - a)]\log[(b - x)/(a - x)]$ $- [1/k(b - a)]\log(b/a)$ or $k(b - a)t = \log\{(a/b)[(b - x)/(a - x)]\}$ or $e^{k(b-a)t} = (a/b)[(b - x)/(a - x)]$. Solve for x. This gives the text answer.
4. To find the inflection point we may as well start with $dP/dt = kP(L - P)$ $= kLP - kP^2$. To find d^2P/dt^2, treat P as a function of t. Then d^2P/dt^2 $= kL\dot{P} - 2kP\dot{P}$. This is 0 when $P = L/2$.
5. The original assumption leads, when fractions are cleared, to $x^2 + 1$ $= A(x + 1) + B(x - 1)$. This is impossible because one side is quadratic and the other linear. If we divide $x^2 - 1$ into $x^2 + 1$ we get $1 + 2/(x^2 - 1)$. We may now apply partial fractions to $2/(x^2 - 1)$.

6. (a) The method of partial fractions applied to $1/(a^2-x^2)$ leads to $\frac{1}{a^2-x^2} = \frac{1}{2a(a-x)} + \frac{1}{2a(a+x)}$. Hence if $dy/dx = 1/(a^2-x^2)$, $y = (1/2a)\log[(a+x)(a-x)]$. If we have y as a function of u and u as a function of x then by the chain rule we get (1): $y = (1/2a))\log[(a+u)/(a-u)]$. To apply this result to $dt/dv = 1/(32-kv^2)$, let $u = \sqrt{k}\,v$ and write $dt/dv = (1/\sqrt{k})\sqrt{k}/(32-u^2)$. We may now integrate by using (1) so that

$$t = \frac{1}{2\sqrt{32k}}\log\frac{\sqrt{32}+\sqrt{k}\,v}{32-\sqrt{k}\,v} + C.$$

Since $v = 0$ when $t = 0$, $C = 0$. If we now solve for v we get the formula in the text.

(b) To obtain the result of 6(b) from that of 6(a) multiply the numerator and denominator of the result in 6(a) by $e^{-\sqrt{32k}\,t}$ and apply the definition of $\tanh x$.

7. Let x represent the amount of substance undissolved after t hours. At this time the concentration in the water will be $(30-x)/100$. Hence $dx/dt = kx[(50/100)-(30-x)/100] = kx[(x+20)/100)$. Then $dt/dx = (1/k)[(1/x)100/(x+20)]$. By using partial fractions we find that $100/x(x+20) = (5/x)-5/(x+20)$. Hence we can integrate and obtain log functions. Further when $t = 0$, $x = 30$ and when $t = 2$, $x = 20$, so that $k = (5/2)\log(5/6)$. Then when $t = 5$, $x = (5/2)\log(5/6) \sim 12$. Hence 18 grams of the original 30 were dissolved.

8. (a) The given information tells us that $N'(t) = kN(t)[M-N(t)]$. As in other such problems we can work with dt/dN and use the method of partial fractions. However, for variety, we introduce the trick of letting $N = 1/\omega$, so that in $kt = \int dN/N(M-N)$ we make the change of variable $N = 1/\omega$, $dN = (-1/\omega^2)d\omega$ and we have $-kt = \int d\omega/(M\omega-1)$ or $-kt = (1/M)\log(M\omega-1)$ or $\omega = (1+De^{-Mkt})/M$. Then $N = M/(1+De^{-Mkt})$.

(b) To find when $N'(t)$ is a maximum we can use the usual max. and min. method. However, the work is extensive. In the present case we use $N'(t)$ to obtain $N''(t) = -kNN'+kMN'-kNN' = kMN'-2kNN' = kN'(M-2N)$. Hence $N'' = 0$ when $N = M/2$ or half the population.

9. The given information amounts to $dP/dt = .05P-.02P^2$. Change to dt/dP and use the method of partial fractions. This yields $P = .05e^{.05t}/(C+.02e^{.05t})$

10. From the given dx/dt we have $1/x(N-x) = k(dt/dx)$. Use of partial fractions gives $(1/Nx)+1/N(N-x) = kdt/dx$. Integration gives $(1/N)\log[N/(N-x)] = kt+C$ or $x = DNe^{Nkt}/(1+De^{Nkt})$ where $D = e^{NC}$. Since $x = x_0$ at $t = 0$ then $x_0 = DN/(1+D)$ or $D = x_0/(N-x_0)$. Then $x = x_0Ne^{Nkt}/(N-x_0+x_0e^{Nkt})$. As t becomes infinite only the exponential terms increase and the limit is x_0N/x_0 or N.

CHAPTER 14, SECTION 4, SECOND SET

1. (a) Let $x/[(x-3)(x+1)^2] = A/(x-3) + B/(x+1) + C/[(x+1)^2]$. By clearing fractions $x = A(x+1)^2 + B(x-3)(x+1) + C(x-3)$. Let $x = -1$, 3, and 0 in turn. Then $C = 1/4$, $A = 3/16$, and $B = -3/16$. Then the given integral becomes $\frac{3}{16}\int dx/(x-3) - \frac{3}{16}\int dx/(x+1) + \frac{1}{4}\int dx/[(x+1)^2]$. Each integral is immediately integrable and the result is in the text.

(b) Let $(3x+2)/x^2(x+1) = A/x + B/x^2 + C/(x+1)$. Clear of fractions. Then $3x + 2 = Ax(x+1) + B(x+1) + Cx^2$. Let $x = 0, -1$, and 1 in turn. Then $B = 2$, $C = -1$, $A = 1$. Then the given integral equals

$$\int dx/x + 2\int dx/x^2 - \int dx/(x+1) = \log x - 2/x - \log(x+1)$$

$$= \log[x/(x+1)] - 2/x + C.$$

(c) Let $(x + 1)/[x(x - 1)^3] = A/x + B/(x - 1) + C/[(x - 1)^2] + D/[(x - 1)^3]$. Clear of fractions. Then $x + 1 = A(x - 1)^3 + Bx(x - 1)^2 + Cx(x - 1) + Dx$. Let $x = 0, 1, -1$, and 2 in turn. Then $A = -1$, and $D = 2$. For $x = -1$ we get $0 = -8A - 4B + 2C - D$ and for $x = 2$ we get $3 = A + 2B + 2C + 2D$. Knowing A and D we can solve the last two equations for B and C. Then $B = 1$ and $C = -1$. Then the given integral equals

$-\int dx/x + \int dx/(x - 1) - \int dx/[(x - 1)^2] + 2 \int dx/[(x - 1)^3]$. Each of these is immediately integrable and we get the text's answer.

(d) Let $(x + 1)/[(x - 2)^2(x - 3)^2] = A/(x - 2) + B/[(x - 2)^2] + C/(x - 3) + D/[(x - 3)^2]$. Clear of fractions. Then $x + 1 = A(x - 2)(x - 3)^2 + B(x - 3)^2 + C(x - 2)^2(x - 3) + D(x - 2)^2$. Let $x = 2$. Then $B = 3$. Let $x = 3$. Then $D = 4$. Let $x = 0$. Then $1 = -18A + 9B - 12C + 4D$. Let $x = 1$. Then $2 = -4A + 4B - 2C + D$. Knowing B and D, solve these last two equations for A and C. Then $A = -7$ and $C = -7$. Then the given integral becomes $7 \int dx/(x - 2) + 3 \int dx/[(x - 2)^2] - 7 \int dx/(x - 3) + 4 \int dx/[(x - 3)^2]$. Each of these immediately integrable. The result is $7 \log[(x - 2)/(x - 3)] - [3/(x - 2)] - [4/(x - 3)] + C$.

CHAPTER 14, SECTION 4, THIRD SET

1. (a) Let $1/x(x^2 + 4) = A/x + (Bx + C)/(x^2 + 4)$. Then $1 = A(x^2 + 4) + Bx^2 + Cx$. Let $x = 0$. Then $A = 1/4$. Let $x = 1$. Then $1 = 5A + B + C$. Let $x = -1$. Then $1 = 5A + B - C$. Since we know A we can solve the last two equations for B and C. Then $B = -1/4$ and $C = 0$. The given integral becomes $1/4 \int dx/x - 1/4 \int [x/(x^2 + 4)]dx = 1/4 \log x - 1/8 \log(x^2 + 4) + C$.

   This can be written $1/4 \log x - 1/4 \log(x^2 + 4)^{1/2} + C$, which gives the text's answer.

   (b) Let $x/(x + 1)(x^2 + 1) = A/(x + 1) + (Bx + C)/(x^2 + 1)$. Then $x = Ax^2 + A + Bx^2 + Bx + Cx + C$. Let $x = -1$. Then $A = -1/2$. Let $x = 0$. Then $0 = A + C$. Let $x = 1$. Then $1 = 2A + 2B + 2C$. Then $C = 1/2$ and $B = 1/2$. The given integral becomes

   $-1/2 \int dx/(x + 1) + 1/2 \int [(x + 1)/(x^2 + 1)]dx = -1/2 \log(x + 1) + 1/2 \int [x/(x^2 + 1)]dx + 1/2 \int dx/(x^2 + 1) = -1/2 \log(x + 1) + 1/4 \log(x^2 + 1) + 1/2 \arctan x + C$.

(c) Let $(x + 1)^2/(x^3 + x) = A/x + (Bx + C)/(x^2 + 1)$. Clear of fractions and let $x = 0$, 1 and -1 to determine A, B and C. The integrations are immediate and give the text's answer.

(d) Let $x^2/(x^4 + 5x^2 + 4) = (Ax + B)/(x^2 + 1) + (Cx + D)/(x^2 + 4)$. Clear of fractions and let $x = 0, 1, -1$ and 2 to determine A, B, C and D. This gives four equations in the four unknowns. We find that $A = 0$, $B = -\frac{1}{3}$, $C = 0$, $D = \frac{4}{3}$. The given integral becomes

$$-\tfrac{1}{3} \int dx/(x^2 + 1) + \tfrac{4}{3} \int dx/(x^2 + 4) = -\tfrac{1}{3} \arctan x + \tfrac{1}{3} \int dx/[(x^2/4) + 1]$$

$$= -\tfrac{1}{3} \arctan x + \tfrac{2}{3} \int \tfrac{1}{2} dx/[(x^2/4) + 1] = -\tfrac{1}{3} \arctan x + \tfrac{2}{3} \arctan x/2 + C.$$

(e) Let $1/(x - 1)(x^2 + 4x + 5) = A/(x - 1) + (Bx + C)/(x^2 + 4x + 5)$. Then $1 = A(x^2 + 4x + 5) + (Bx + C)(x - 1)$. Let $x = 1$. Then $A = \frac{1}{10}$. Let $x = 0$. Then $1 = 5A - C$ and $C = -\frac{1}{2}$. Let $x = -1$. Then $1 = 2A + 2B - 2C$. Then $B = -\frac{1}{10}$. Hence the given integral becomes

$\frac{1}{10} \int dx/(x - 1) + \int [(-\frac{1}{10}x - \frac{1}{2})/(x^2 + 4x + 5)]dx$. To handle the second integral complete the square in the denominator or

$\int \{(-\frac{1}{10}x - \frac{1}{2})/[(x + 2)^2 + 1]\} dx$. Let $u = x + 2$ and so $x = u - 2$. Then the integral becomes (since $dx = du$)

$$\int [(-\tfrac{1}{10}u + \tfrac{1}{5} - \tfrac{1}{2})/(u^2 + 1)] du = -\tfrac{1}{10} \int [u/(u^2 + 1)] du - \tfrac{3}{10} \int du/(u^2 + 1)$$

$= -\frac{1}{20} \log(u^2 + 1) - \frac{3}{10} \arctan u + C$. Replace u by $x + 2$ and add back the integral neglected above. This gives the text's answer.

(f) Let $1/(x - 1)^2(x^2 + 4x + 5) = A/(x - 1) + B/(x - 1)^2 + (Cx + D)/(x^2 + 4x + 5)$ or $1 = A(x - 1)(x^2 + 4x + 5) + B(x^2 + 4x + 5) + (Cx + D)(x - 1)^2$. Let $x = 1$. Then $B = \frac{1}{10}$. Let $x = 0, -1$ and 2 in turn which gives three equations in A, B, C and D. Since we know B, we can find A, C and D. The integral of $(Cx + D)/(x^2 + 4x + 5)$ is handled by the method of the text or of (e) just above. The result is $-\frac{3}{50} \log(x - 1) - \frac{1}{10}[1/(x - 1)] + \frac{3}{100} \log(x^2 + 4x + 5) + \frac{2}{25} \arctan(x + 2) + C$.

(g) Let $1/(x^3 + 1) = A/(x + 1) + (Bx + C)/(x^2 - x + 1)$. Then $1 = A(x^2 - x + 1) + (Bx + C)(x + 1)$. Let $x = -1$. Then $A = \frac{1}{3}$. Let $x = 0$. Then $1 = A + C$. Then $C = \frac{2}{3}$. Let $x = 1$. Then $1 = A + 2B + 2C$. Then

$B = -\frac{1}{3}$. Thus the given integral becomes $\frac{1}{3} \int dx/(x + 1)$

$+ \int [(-\frac{1}{3}x + \frac{2}{3})/(x^2 - x + 1)]dx$. Complete the square in the second integral. This integral becomes $-\frac{1}{3} \int \{(x - 2)/[(x - \frac{1}{2})^2 + \frac{3}{4}]\} dx$. Let

$u = x - \frac{1}{2}$, so that $x = u + \frac{1}{2}$. Then the integral becomes

$$-\tfrac{1}{3} \int [(u + \tfrac{1}{2} - 2)/(u^2 + \tfrac{3}{4})] du = -\tfrac{1}{3} \int [u/(u^2 + \tfrac{3}{4})] du + \tfrac{4}{6} \int du/(\tfrac{4}{3}u^2 + 1)$$

$$= -\tfrac{1}{6} \int [2u/(u^2 + \tfrac{3}{4})] du + (\tfrac{4}{6})(\sqrt{3}/2) \int [(2/\sqrt{3}) du]/(\tfrac{4}{3}u^2 + 1)$$

$= -\frac{1}{6} \log(u^2 + \frac{3}{4}) + (1/\sqrt{3}) \arctan(2u/\sqrt{3})$. Replace u by its value and bring in the neglected integral and we get the text's answer.

CHAPTER 14, SECTION 4, FOURTH SET

1. (a) We can apply at once the reduction formula at the top of the page and write down the text's answer.

   (b) Let $2x/(1+x)(1+x)^2 = A/(1+x) + (Bx+C)/(1+x^2) + (Dx+E)/(1+x^2)^2$. Then $2x = A(1+x^2)^2 + (Bx+C)(1+x)(1+x^2) + (Dx+E)(1+x)$. Let $x = -1$. Then $-2 = 4A$ and $A = -\frac{1}{2}$. Let $x = 0$. Then $A + C + E = 0$. Let $x = 1$. Then $2 = 4A + 4B + 4C + 2D + 2E$. Let $x = 2$. Then $4 = 25A + 30B + 15C + 6D + 3E$. Let $x = -2$. Then $25A + 10B - 5C + 2D - E = -4$. If we solve for B, C, D and E and we get $A = -\frac{1}{2}$, $B = \frac{1}{2}$, $C = -\frac{1}{2}$, $D = 1$, $E = 1$. The integrations are all straightforward except for $\int dx/(x^2+1)^2$. This is integrated by letting $x = \tan u$. The entire result is $\frac{1}{4}\log[(x^2+1)/(x+1)^2] + (x-1)/2(x^2+1) + C$.

CHAPTER 14, SECTION 5

1. (a) This integrand is a rational function of $\cos\theta$. Hence let $\theta = 2\arctan u$. Then, by (43), $\cos\theta = (1-u^2)/(1+u^2)$ and, by (45), $d\theta/du = 2/(1+u^2)$. Then

$$\frac{d\theta}{5+4\cos\theta} = \frac{2\,du}{1+u^2}\,\frac{1}{5+4[(1-u^2)/(1+u^2)]} = \frac{2\,du}{9+u^2}$$

$$= \frac{2}{3}\,\frac{\frac{1}{3}}{1+(u/3)^2}. \text{ Then } \int\frac{d\theta}{5+4\cos\theta} = \frac{2}{3}\int\frac{\frac{1}{3}du}{1+(u/3)^2} = \frac{2}{3}\arctan(u/3) + C.$$

   Since $u = \tan(\theta/2)$ we have the text's result.

   (b) Let $x = 2\arctan u$. Then using (43), (44) and (45) we have $\int dx/(1+\sin x+\cos x) = \int du/(1+u) = \log(1+u) + C$. Now $u = \tan(x/2)$. Hence the result $\log[1+\tan(x/2)] + C$.

   (c) Using the change of variable (42), (44) for $\sin x$, and (44) and (43) for $\tan x$ we have $\int dx/(\sin x+\tan x) = \int[(1/2u - u/2)]du = \frac{1}{2}\log u - u^2/4 + C$. Substituting the value of u gives the text's result.

CHAPTER 14, SECTION 6

1. (a) Use formula #23 followed by the second form in formula #21.

   (b) Use formula #64. Ans. $(2/\sqrt{3})\arctan[(2x+1)/\sqrt{3}] + C$.

   (c) Use formula #22 followed by the first form in formula #21.

   (d) Let $x = \sqrt{10}\sin\theta$; then $(dx/du)\,du = \sqrt{10}\cos\theta\,d\theta$.

$\int (\sqrt{10 - x^2}/x^4)\,dx = {}^1\!/_{10} \int (\cos^2\theta/\sin^4\theta)\,d\theta = {}^1\!/_{10} \int [(1 - \sin^2\theta)/\sin^4\theta]\,d\theta$
$= {}^1\!/_{10} \int (1/\sin^4\theta)\,d\theta - {}^1\!/_{10} \int (1/\sin^2\theta)\,d\theta$. Now $\int 1/\sin^4\theta$ can be evaluated by applying formula #72 in reverse. That is, solve for the integral on the right side and then apply it once. Thus
$\int \sin^{n-2} u\,du = [n/(n-1)] \int \sin^n u\,du + [1/(n-1)] \sin^{n-1} u \cos u$. In our case $n - 2 = -4$ so that $n = -2$. Then ${}^1\!/_{10} \int \sin^{-4}\theta\,d\theta = -{}^1\!/_{10}(\cos\theta/3\sin^3\theta)$
$- {}^2\!/_{10}(\cos\theta/3\sin\theta)$. As for the second integral, $-{}^1\!/_{10} \int (1/\sin^2\theta)\,d\theta$
$= -{}^1\!/_{10} \int \csc^2\theta = +{}^1\!/_{10}\cot\theta$. Now take the sum of these last two results and change back to x. Simplification of the sum of terms in x gives $[-(10 - x^2)^{3/2}/30x^3] + C$.

(e) Use form #23 followed by the upper form in #21.
(f) Use form #52. Ans. ${}^1\!/_3 \log[(\sqrt{x^2 + 9} - 3)/x] + C$.
(g) Factor out 3 from the radical and apply #51.

Solutions to Chapter 15

CHAPTER 15, SECTION 2

1. The formula for volume of revolution is $V = \int_a^b \pi y^2\,dx$. Hence $V = \int_0^5 (\pi x^2/9)\,dx = 125\pi/27$.
2. $V = \int_3^5 \pi 16x\,dx = 128\pi$.
3. $V = \int_0^2 \pi y^2 dx = \pi\int_0^2 8x dx = 4\pi x^2\Big|_0^2 = 16\pi$.
4. The y-values when $x = 2$ are -4 and 4. The volume in question is the difference between the volume generated by a rectangle and the volume generated by an arc of $y^2 = 8x$. $V = \int_{-4}^{4} \pi(2)^2 dy - \int_{-4}^{4} \pi x^2 dy = \int_{-4}^{4} \pi(4-y^4/64) = 128/5\pi$.
5. $V = \int_2^4 \pi(25 - x^2)\,dx = 94\pi/3$.
6. $V = 2\int_0^4 \pi[8 - (x^2/2)]\,dx = 128\pi/3$.
7. $V = 2\int_0^{2\sqrt{2}} \pi(16 - 2x^2) = 128\sqrt{2}\pi/3$.
8. The line $y = 3$ intersects the hyperbola at $x = 4\sqrt{2}$, and the hyperbola cuts the x-axis at $x = 4$. The required volume is the difference of the volume of the cylinder of height $4\sqrt{3}$ and radius 3 and the volume generated by revolving the hyperbola from $x = 4$ to $x = 4\sqrt{2}$. Thus $V = \pi 3^2 4\sqrt{2}$ $-\int_4^{4\sqrt{2}} \pi[(9x^2 - 144)/16]\,dx$. $V = 24\pi(2\sqrt{2} - 1)$.

CHAPTER 15, SECTION 3

1. The volume generated by revolving about the x-axis is $V = \int_0^3 \pi(9 - x^2)^2\,dx$ $= 486\pi/5$. The volume generated by revolving around the y-axis can be done by the shell method. Here the formula is $V = \int_a^b 2\pi xy\,dx$. In our case $V = \int_0^3 2\pi x(9 - x^2)\,dx = 81\pi/2$. The second volume can be done by considering

the volume generated by the curve $x = \sqrt{9-y}$ from $y = 0$ to $y = 9$ in a revolution around the y-axis. Hence we merely rewrite the usual formula to read $V = \int_c^d \pi x^2\,dy$.

2. The area lies above the arc of $y = (12 - x^3)/8$ and below the line $y = 2$. The line cuts the arc at $x = -\sqrt[3]{4}$. Hence the y-values in the area are given by $2 - [(12 - x^3)/8]$ or $\frac{1}{2} + x^3/8$. Using the cylindrical shell method we have $V = \int_{-\sqrt[3]{4}}^{0} 2\pi x(\frac{1}{2} + x^3/8)\,dx$. The result is $-3\pi\sqrt[3]{2}/5$. The negative sign appears in the answer because we are working with a negative x, a positive dx and positive y-values. The sign can be ignored.
3. The cylindrical shell method can be used. We take the axis of revolution to be the y-axis and the equation of the circle to be $(x - b)^2 + y^2 = a^2$. Then $V = \int_{b-a}^{b+a} 2\pi x\sqrt{a^2 - (x-b)^2}\,dx$. Let $u = x - b$ or $x = u + b$. Then $V = \int_{-a}^{+a} 2\pi(u + b)\sqrt{a^2 - u^2}\,du = \int_{-a}^{+a} 2\pi u\sqrt{a^2 - u^2}\,du + \int_{-a}^{+a} 2\pi b\sqrt{a^2 - u^2}\,du$. The first integral is immediately integrable. For the second use formula #26 in the Table. Then $V = 2\pi^2 a^2 b$.
4. Use the cylindrical shell method. Then $V = \int_0^{\pi} 2\pi x \sin x\,dx$. One can integrate by parts or use formula #82 of the Table with $n = 1$. The result is $2\pi^2$.
5. By elementary geometry the radius of the cylindrical hole is $\sqrt{b^2 - 9}$. The remaining volume is a figure of revolution. Taking axes in the usual way with origin at the center of the sphere the upper half of the volume is generated by an arc of $x^2 + y^2 = b^2$ extending from $x = \sqrt{b^2 - 9}$ to $x = b$. Then $V/2 = \int_{\sqrt{b^2-9}}^{b} 2\pi xy\,dx = \int_{\sqrt{b^2-9}}^{b} 2\pi x\sqrt{b^2 - x^2}\,dx = 18\pi$. The answer is remarkable because the result is independent of the radius b of the sphere. As long as $b > 3$, the sphere can have a hole 6 inches deep drilled in it. If b is very large the radius of the cylindrical hole is also very large so that the remaining volume is constant.
6. The spherical shell method is best because the density varies with the radius. The mass of the i-th shell is $2\pi\rho r_i^2\Delta r$. Then the integral (cf. p.) is $\int_0^R 4\pi\rho r^2\,dr$, where R is the radius of the earth.
7. Use the spherical shell method and the idea of Exercise 6 except that ρ is now r^2. Hence mass $M = \int_0^R 4\pi r^4\,dr = 4\pi R^5/5$.
8. Use the cylindrical shell method. Here $y = \sqrt{a^2 - (x-b)^2}$. Then $V = 2\int_0^{a+b} 2\pi x\sqrt{a^2 - (x-b)^2}\,dx$. To integrate let $u = x - b$ or $x = u + b$. Then $V = 4\pi\int_{-b}^{a} u\sqrt{a^2 - u^2}\,du + 4\pi\int_{-b}^{a} b\sqrt{a^2 - u^2}\,du$. The first integral is immediately integrable and the second is done by formula #26 in the Table. The result is $(4\pi/3)(a^2 - b^2)^{3/2} + 2\pi b[(\pi a^2/2) + b\sqrt{a^2 - b^2} + a^2\sin^{-1}(b/a)]$.

CHAPTER 15, SECTION 4

1. (a) We use (24). Then $s = \int_0^4 \sqrt{1 + (9x/4)}\,dx$.

   (b) It is easier to write $y = \cosh x$. Then $y' = \sinh x$. $s = \int_{-1}^{1} \sqrt{1 + \sinh^2 x}\,dx$

   $= \int_{-1}^{1} \cosh x\,dx = \sinh x \Big|_{-1}^{1}$. Since $\sinh x = (e^x - e^{-x})/2$ we get $e - e^{-1}$.

   (c) This problem can be done by using (24). The integral to which it leads can be evaluated by a trigonometric change of variable. However we can also use (29). In this case $x = y^{3/2}$ and we have a problem of the type done in (a) but with 0 and 4 as the end values for the y-interval. We obtain the same definite integral as in (a) but with x and y interchanged.

   (d) $y' = x^2/2 - 1/2x^2$. Hence $s = \int_1^3 \sqrt{1 + (x^2/2 - 1/2x^2)^2}\,dx$. Squaring and simplifying the radicand gives a perfect square so that the integral reduces to $\int_1^3 (x^2/2 + 1/2x^2)\,dx$. Ans. $^{14}/_3$.

   (e) Since $y' = \tan x$, hence $s = \int_0^{\pi/3} \sqrt{1 + \tan^2 x}\,dx = \int_0^{\pi/3} \sec x\,dx$

   $= \log(\sec x + \tan x)\Big|_0^{\pi/3} = \log(2 + \sqrt{3})$.

   (f) $y' = x^3 - (1/4x^3)$; $\sqrt{1+y'^2} = x^3 + (1/4x^3)$. $s = \int_1^3 (x^3 + 1/4x^3)\,dx$
   $= (x^4/4) - (1/8x^2)\,\Big|^3 = 20\,{}^1\!/9$.

   (g) $y = (2/3\sqrt{a})x^{1/2}$; ${}^1y' = (1/\sqrt{a})x^{1/2}$; $\sqrt{1+y'^2} = (x/a)+1$.
   $s = \int_0^x \sqrt{(x/a)+1}\;dx = (2a/3)\{(1+x/a)^{3/2} - 1\}$.

2. Since $y' = x/8$ we are led at once to an integral of the form given in the Exercise. The result follows at once.

3. By letting $x = r\sin\theta$ and $dx = r\cos\theta\,d\theta$ we obtain from (33)

   $$s = \int_0^{\pi/2} (r^2\cos\theta / r\cos\theta)\,d\theta = r\theta\Big|_0^{\pi/2} = \pi r/2.$$

CHAPTER 15, SECTION 5

1. For a straight line the inclination of the tangent is a constant angle. Hence $d\phi/ds = 0$.
2. The central angle θ of the circle and the angle of inclination ϕ of the tangent are related by $\phi = \theta + \pi/2$ in the first quadrant (and by minor variations of this in other quadrants). Moreover $s = R\theta$ so that $\theta = s/R$. Hence $\phi = s/R + \pi/2$. Then $d\phi/ds = 1/R$.
3. The line can be represented by $y = mx + b$. Then $y'' = 0$ and so (42) will yield 0.
4. Let the upper half of the circle by $y = \sqrt{R^2 - x^2}$. Then $y' = -x(R^2 - x^2)^{-1/2}$ and $y'' = -R^2/(R^2 - x^2)^{3/2}$. If we substitute in (42) we obtain $1/R$.
5. Since $y'' = 0$ at a point of inflection, $K = 0$.
6. In this Exercise we want $1/K$ where $y' = 0$ on the curve. Hence, by (42), $K = y''$ and $R = 1/K = 1/y''$.
   (a) Since $y'' = 2/a$, $R = a/2$.
   (b) Same work and result as (a).
   (c) We calculate y' and y''. Since $y' = 0$ at $x = 0$, $y'' = 1/a$ at $x = 0$ and $R = a$.
   (d) Here $y' = 0$ when $x = 0$. At $x = 0$, $y'' = b/a^2$. Hence $R = a^2/b$.
   (e) We work with the lower half of the ellipse. $y' = 0$ at $x = 0$. $y'' = b/a^2$ at $x = 0$ and so $R = a^2/b$.
7. If $a = b$, $R = a$ as it should be.
8. If we take the ellipse in the standard position with the foci at $(c, 0)$ and $(-c, 0)$ we encounter the difficulty that at $(a, 0)$ $y' = \infty$. However if we calculate the curvature, using (42), at any point (x, y) of the ellipse it turns out that at $(a, 0)$ the infinity caused by y being 0 at $(a, 0)$ is offset and the result for the curvature is $-a/b^2$. Alternatively we can compute the curvature for the ellipse $(x^2/b^2) + (y^2/a^2) = 1$ whose foci are $(0, c)$ and $(0, -c)$. In either case $R = b^2/a$ and this is half the latus rectum.

CHAPTER 15, SECTION 6

1. Here $y = \sqrt{a^2 - x^2}$. Then, by (48), $S = 2\pi \int_c^d a\,dx = 2\pi a(d - c)$.
2. (a) By (48), $S = 4\pi \int_0^\infty \sqrt{x+1}\,dx = 208\pi/3$.
   (b) By (48), $S = (\pi/6) \int_0^2 \sqrt{1 + x^4}\, 4x^3\,dx = (\pi/9)(17\sqrt{17} - 1)$.
   (c) One can work with the exponential expression or let $y = \cosh x$. Then $S = 2\pi \int_{-1}^1 \cosh x\sqrt{1 + \sinh^2 x}\,dx = 2\pi \int_{-1}^1 \cosh^2 x\,dx$. Now replace $\cosh x$ by its value. The result is in the text.

(d) $S = 2\pi \int_0^{\pi} \sin x\sqrt{1 + \cos^2 x}\,dx$. Let $u = \cos x$. To make the substitution (as opposed to change of variable) we must have $(du/dx)\,dx$ in the integrand. Now $du/dx = -\sin x$. But we have $S = -2\pi \int_0^{\pi} \sqrt{1 + \cos^2 x}\,(-\sin x)\,dx$
$= -2\pi \int_1^{-1} \sqrt{1 + u^2}\,(du/dx)\,dx = -2\pi \int_1^{-1} \sqrt{1 + u^2}\,du$
$= -2\pi[(u/2)\sqrt{u^2 + 1} + \tfrac{1}{2}\log|u + \sqrt{u^2 + 1}|]\Big|_1^{-1}$. The result is given in the text.

3. This just calls for replacing x by y and y by x in (48). Then $S = 2\pi \int_c^d x\sqrt{1 + \dot{x}^2}\,dy$ where x is a function of y.

4. (a) We use the formula given in Exercise 3. Since $x' = 3y^2$ we have $S = 2\pi \int_0^3 y^3\sqrt{1 + 9y^4}\,dy$. Let $u = 1 + 9y^4$ and the integration is immediate since we have the du/dx except for a constant factor.

   (b) Using the formula in Exercise 3 we get $S = (2\pi/3) \int_0^8 y^{1/3}\sqrt{9y^{2/3} + 4}\,dy$. To integrate let $y^{1/3} = \tfrac{2}{3}\tan\theta$. This change of variable transforms the integral (apart from a constant factor) to $\int \tan^3\theta \sec^3\theta\,d\theta$. Write this (using $\tan^2\theta = \sec^2\theta - 1$) as $\int \sec^4\theta \sec\theta\tan\theta\,d\theta - \int \sec^2\theta \sec\theta\tan\theta\,d\theta$ and let $u = \sec\theta$ in each case. Since the end values for y are 0 and 8 the end values for θ are 0 and $\tan^{-1} 3$.

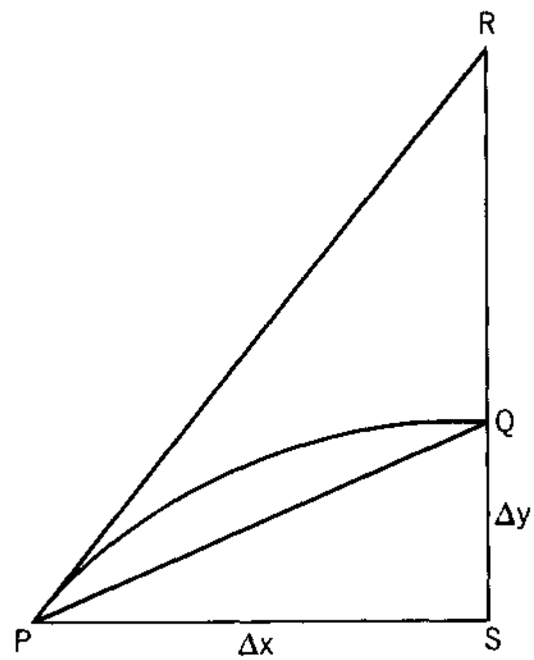
   (c) Using the formula in Exercise 3 we get $S = 4\pi \int_0^4 \sqrt{y + 1}\,dy$. Then $S = (8\pi/3)(5^{3/2} - 1)$.

CHAPTER 15, SECTION 7

1. We may use the criterion in the text. If the difference in the approximating elements, apart from the factor Δx, approaches 0 as Δx does, then the difference will not contribute to the definite integral. In the present case the difference is $\Delta y_i \Delta x/2$ and, apart from the factor Δx, the quantity $\Delta y_i/2$ approaches 0 as Δx does. Hence this difference does not contribute to the integral.

2. If we used the cylinder the surface area is $2\pi y_i \Delta x$. In the text on p.   it is shown that we can take $2\pi\bar{y}_i\sqrt{1+\bar{y}_i'^2}\,\Delta x$ as the approximating element, where $\bar{y}_i$ and $\bar{y}_i'$ are taken at the same value of x_i. Let us drop the bars for the sake of this Exercise. Now we consider $2\pi y_i\sqrt{1+y_i'^2}\,\Delta x - 2\pi y_i \Delta x$ $= 2\pi y_i \Delta x(\sqrt{1+y_i'^2} - 1)$. This difference, apart from the factor Δx, does not approach 0 as Δx does because y_i' is the slope of the curve and this does not approach 0.

3. The lateral surface of a (right circular) cone of radius r and altitude h is $\pi r\sqrt{r^2 + h^2}$. Hence the answer obtained by the process is not correct. The reason is precisely the point of Exercise 2. Cylinders as approximating elements to obtain a surface of revolution are not good enough. One must use truncated cones as the text proper does.
4. (a) Each approximation, like the one in the figure, gives $b + h$ and so $\lim_{n\to\infty} S_n = b + h$.
   (b) The result $b + h$ is incorrect as the Pythagorean Theorem tells us.
   (c) The approximating elements, e.g., $CC_1 + C_1D_1$ as an approximation to CD_1, are not good enough.
   (d) It is true that the points of the approximating elements do approach the points of CB. But the segments and their lengths are two different entities.
5. We may replace y_i' by y_i because $(2y_i - y_i') - (2y_i - y_i) = y_i - y_i'$ and as Δx_i approaches 0 $y_i - y_i'$ must approach 0, for, y_i and y_i' are y-values in the same subinterval Δx_i and as Δx_i approaches 0 there y-values must approach each other. Then $S_n = y_1\Delta x_1 + y_2\Delta x_2 + \cdots + y_n\Delta x_n$ and $\lim_{n\to\infty} S_n$ $= \int_1^4 y\,dx = \int_1^4 3x^2\,dx = 63$.
6. (a) PR is the slope of the tangent to the curve at P. Then $y' = RS/PS$ or since $PS = \Delta x$, $RS = y\Delta x$. Then $PR = \sqrt{1+y'^2}\,\Delta x$. Since dx is just another symbol for Δx, PR is ds. Arc PQ is the actual change in arc length from P to Q so that Δs = arc PQ. Chord $PQ = \sqrt{(\Delta x)^2 + (\Delta y)^2}$.

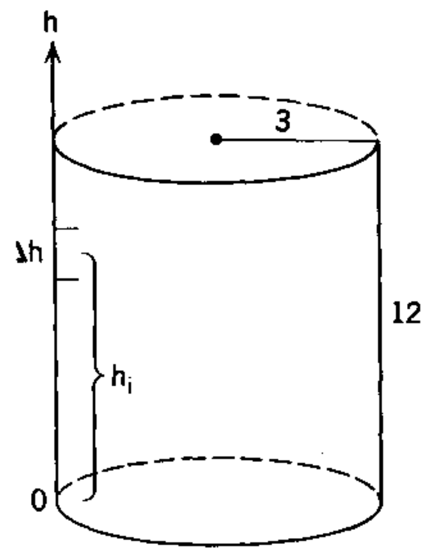


   (b) $ds = PR$ and $\sqrt{(\Delta x)^2 + (\Delta y)^2} = PQ$. Now $PR - PQ < RQ$ because the difference of two sides of a triangle is less than the third side. Now as Δx approaches 0, RQ approaches 0. Hence so does $PR - PQ$. Then PR may replace PQ in a summation leading to a definite integral.

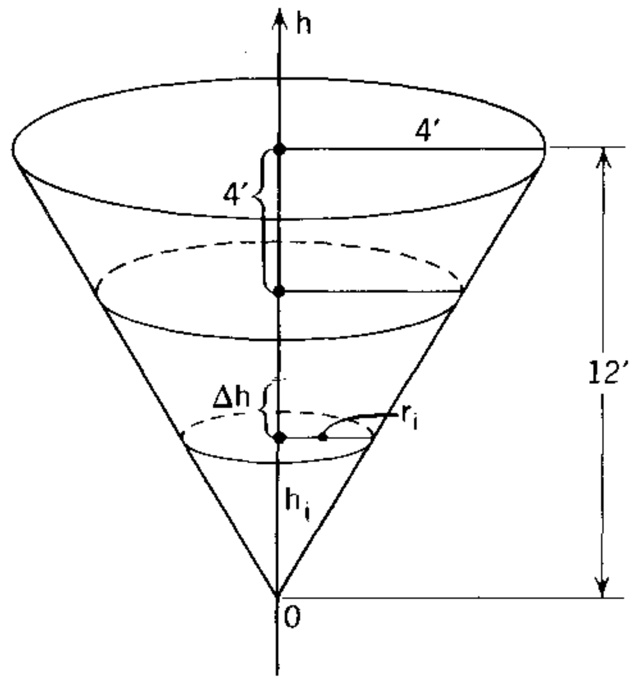
Solutions to Chapter 16

CHAPTER 16, SECTION 2

1. We apply (5) except for the numbers involved. In our case $W = \int_{4000}^{4300}[(GM \cdot 1000)/r^2]\,dr$. By the fundamental theorem $W(r) = -1000\,GM/r$ and $W = -1000\,GM/r\,\Big|_{4000}^{4300}$. Converting to feet and using the value of $GM = 32(4000 \cdot 5280)^2$ we get $4.73 \cdot 10^{10}$ ft.-pdl.
2. To lift the 1000 pound object from 300 miles to 600 miles should require less work than in Exercise 1 because the force of gravity is weaker at higher altitudes. We have this time $W = -1000\,GM/r\,\Big|_{4300}^{4600}$. Converting to feet and using the value of GM gives $4.06 \cdot 10^{10}$ ft.-pdl.
3. Since F is just 32m, the work depends only on the difference in the y-values of the two points and not on the particular hill along which the object is pushed. The same amount of work would be involved in merely lifting the object from a height y_1 to a height y_2.
4. Yes. Even if F varies along the curve the argument holds. The value of F is not involved. Of course in evaluating the final integral the result does depend on the value of F but the work will not depend on the path.
5. The argument which leads to (5) still holds even though m is also a function of r. Hence $W = \int_{4000}^{4300}[GM(2000R^3/r^3)/r^2]\,dr = \int_{4000}^{4300}(2000GM\,R^3/r^5)\,dr$. Of course in the calculation all distances must be in feet.
6. The element of volume $\pi \cdot 3^2 \cdot \Delta h$ weighs $(62.5)9\pi\Delta h$ lbs. This must be brought down by gravity a distance h_i feet to the bottom. Then the element of work is $(62.5)9\pi h_i\Delta h$. The sum of all such elements whose Δh's fill out the interval from $h = 0$ to $h = 12$ is the n-th approximation to the work and the work is $W = \int_0^{12}(62.5)9\pi h\,dh$ $= 40{,}500\pi$ ft-lb.



7. Any element of volume of circular cross-section $A(r_i)$ and thickness Δh weighs $80\ A(r_i)\Delta h$ lb. and must be lifted $12 - h_i$ feet to the top of the reservoir. The work done in lifting this element of volume to the top is, then, $80\ A(r_i)\Delta h(12 - h_i)$. At any h_i, by similar triangles, $r_i = h_i/3$. The typical element of work is therefore $80\pi(h_i/3)^2(12 - h_i)\Delta h$. The n-th approximation to the work is the sum of all such elements from $h = 0$ to $h = 8$ and the work is $W = \int_0^8 (80\pi/9)h^2(12 - h)\,dh = 81{,}920\,\pi/9$ ft.-lb.



CHAPTER 16, SECTION 3

1. The total revenue is $R = \int_0^5 100\ dt/(1+t)^2$.
2. One cannot obtain the total income as a sum of the incomes at each instant just as one cannot obtain the area under a curve as a sum of the infinite number of lengths of the line segments which fill out the area.
3. $H = \int_0^{1000} [80-.05(x-1)dx = 5050$ hours.
4. $H = \int_0^{1000} 100\ dx/x^{1/2} = 200x^{1/2}\Big|_0^{1000} = 200\sqrt{1000}$ hours.

CHAPTER 16, SECTION 4

1. If the width is 2ℓ as in Fig. 16-5, then $x = \ell$ and if S is measured from the vertex, then when $x = \ell$, s is, say, S_o. Substitute these values in (19). Then the one unknown is c and in theory it is determined. We can then calculate the sag d from (24).
2. Following the suggestion we have $T^2 = w^2c^2[1+(e^{2x/c}-2+e^{-2x/c})/4] = w^2c^2[(e^{x/c}+e^{-x/c})/2]^2 = w^2y^2$ by (21).
3. According to Exercise 2, the tension at B due to the weight of the cable is w times the y-value of B. Since the portion BC extends to the directrix (which is the x-axis), BC has just the weight to offset exactly the tension at B due to the cable.
4. For any catenary the length LB is given by (19) wherein we may set $x = a$. Hence $LB = c(e^{a/c}-e^{-a/c})/2$. If the cable is not to slip at B the value of h must be equal to the y-value of the curve at B, in view of the result of Exercise 3. Then at $x = a$, h must equal, by (31), $c(e^{a/c}+e^{-a/c})/2$. Then we wish to minimize $2(LB+h) = 2ce^{a/c}$. Now the independent variable is c because the height of the catenary above the directrix is still open. If we call $2(LB+h)$, z, then $dz/dc = 2ce^{a/c}(-a/c^2)+2e^{a/c}$. $dz/dc = 0$ when $c = a$. Then $y = 2ae$.
5. Use (21) for y and (19) for s and the result is immediate.
6. The equation of the tangent at (x_0,y_0) is $y-y_0 = m(x-x_0)$ where m is the slope of the catenary at (x_0,y_0). Since $y = c \cosh(x/c)$, $y' = \sinh(x/c) = \sinh(x_0/c)$ at (x_0,y_0). Then $y-c \cosh(x_0/c) = \sinh(x_0/c)(x-x_0)$. Since the tangent must pass through the origin, $y = 0$ when $x = 0$. Then $-\cosh(x_0/c) = -x_0\sinh(x_0/c)$ which gives the text's result.

7. Since weight is 32 times the mass, the given information is that $\Delta w = w(s)\Delta s = k\Delta x$; $w(s)$ is the weight per unit foot so that $w(s)\Delta s$ is the weight of a length Δs of cable. Then $w(s)\,ds = k\,dx$ and we may replace $w(s)\,ds$ in (36) by $k\,dx$. This replacement is actually a change of variable. Then two integrations with respect to x yield a parabola.
8. We want $A = \int_0^d y\,dx$ where y represents the catenary. The result is $(c^2/2)(e^{d/c} - e^{-d/c})$. The length of arc is given by (19) in which we replace x by d. Hence the area is that of the rectangle described. We can of course write $\sinh(d/c)$ for the exponential expression.
9. The slope, in view of (21), is, $y' = (e^{x/c} - e^{-x/c})/2$. Using (19) we see that the slope is the arc length divided by c.
10. The curvature is given by (42) of the preceding chapter. Hence one merely substitutes in that and then takes the reciprocal.
11. We may use (19) in which $s = 25$ when $x = 15$. This gives an equation for c. But $c = T_0/w$ and we are given w.
12. We may use $y' = (e^{x/c} - e^{-x/c})/2$ in which $x = 50$ and $y' = 3/4$ to obtain a quadratic equation in $e^{50/c}$. Then $50/c = \log 2 = 0.693$ and $c = 71$. Now using (21), $y = 71(e^{50/c} + e^{-50/c})/2 = 71[2 + (1/2)]/2 = 89$. Hence the sag, which is $y - c$, $= 18$ ft. approx.

CHAPTER 16, SECTION 5

1. Introduce an r-axis as in Fig. 16-10, so that the rod extends from $r = 0$ to $r = \ell$ and the mass m is located at $r = \ell + h$. Here the mass per unit length is M/ℓ and so the gravitational force of the i-th subinterval on the mass m is $G\{(M(\ell)\Delta r \cdot m)/[(h + \ell) - r_i]^2\}$ (cf.(40)). Thus, in place of (27), we have $F = \int_0^\ell \{GMm/\ell[(h + \ell) - r]^2\}\,dr$. Using the substitution $u = (h + \ell) - r$ we obtain $F + GMm/h(h + \ell)$.
2. See Fig. 1. By symmetry only a vertical force acts on m. The vertical force exerted by the i-th subinterval of the semicircle is $G(M/\pi a)\Delta s \cdot \sin\theta_i/a^2$. For the circle $\Delta s = a\Delta\theta$. Thus $F = \int_0^\pi (GMm/\pi a^2)\sin\theta\,d\theta = 2GMm/\pi a^2$.

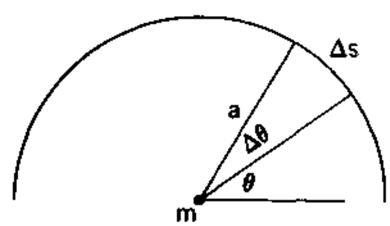


Fig. 1

3. The force of attraction of two particles of masses m and M separated by a distance $a\sqrt{\pi/2}$ is $GMm/a^2(\pi/2) = 2GMm/\pi a^2$. This is the result of Exercise 2.
4. See Fig. 2. Only a vertical force acts on m. The force exerted by the i-th subinterval of the circle is $G(M/2\pi a)m\,\Delta s\cos\theta_i/r^2$. Using $\Delta s = a\Delta\phi$,

$r = \sqrt{h^2 + a^2}$ and $\cos\theta_i = h/r$ we find that $F = (GMm/2\pi)(h/r^3)\int_0^{2\pi} d\phi$, which gives the text's answer.

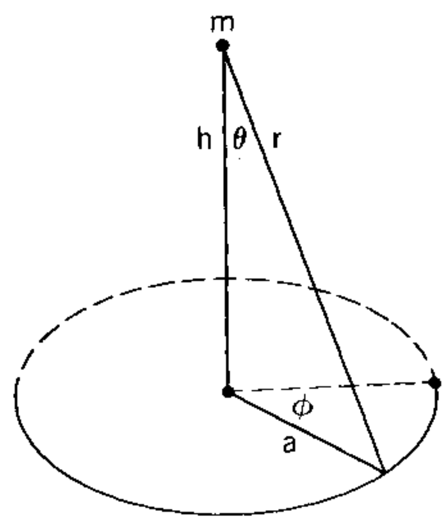


Fig. 2

5. (a) Measure x from Q along QA. Then the horizontal force exerted by the i-th subinterval of AB is $GM \cdot 1\Delta x \sin\theta_i/r_i^2$, where r_i is the distance from P to x_i and θ_i is the angle between PQ and the line from P to x_i. The vertical force is $GM\Delta x \cdot 1\cos\theta_i/r_i^2$. By Fig. 16-12 and our definitions of x_i, r_i and θ_i we have $r_i = p/\cos\theta_i$, $x_i = r_i \sin\theta_i = p\tan\theta_i$. Hence $\Delta x_i = dx_i = p\sec^2\theta_i\Delta\theta$. Hence the horizontal force is $F_H = (GM/p)\int_\beta^\alpha \sin\theta\, d\theta$. The vertical force is $F_V = (GM/p)\int_\beta^\alpha \cos\theta\, d\theta$.

   After integrating use the identity for $\sin A - \sin B$ and for $\cos A - \cos B$ to convert to the answer in the text.

   (b) To get the magnitude of the resultant take $\sqrt{F_H^2 + F_V^2}$. This gives $(2GM/p)\sin[(\alpha - \beta)/2]$. $(\alpha - \beta)/2$ is half the angle APB.

6. According to Exercise 4, the force of a circular wire is $GMmh/(a^2 + h^2)^{3/2}$. For us each wire has mass $(m/\ell)dh$. Moreover we must treat h as variable. There are two shells acting on the unit mass $(m = 1)$. There is the shell from 0 to b pulling upward and the shell from 0 to $\ell - b$ pulling downward. Hence the net force is $(GM/\ell)\int_0^b h\,dh/(a^2 + h^2)^{3/2} - (GM/\ell)\int_0^{\ell - b} h\,dh/(a^2 + h^2)^{3/2}$.

   Let $u = a^2 + h^2$. Then the text result follows at once.

7. When the particle is displaced x units along the axis of the shell it is attracted by a shell of length $\ell - x$ in one direction and by a shell of length $\ell + x$ in the other direction. If we use the result of Exercise 6, then we let $b = \ell - x$ and $\ell - b = \ell + x$. Hence the net force exerted on the unit mass is $(GM/2\ell)[1/\sqrt{a^2 + (\ell - x)^2} - 1/\sqrt{a^2 + (\ell + x)^2}]$. By Newton's second law the negative of this quantity is equal to $\ddot{x}$. Assuming that x is small and therefore x^2 is negligible, we have $\ddot{x} = -GM/2\ell[1/\sqrt{a^2 + \ell^2 - 2\ell x} - 1/\sqrt{a^2 + \ell^2 + 2\ell x}]$. We follow the suggestion replacing $[a^2 + \ell^2 - 2\ell x]^{-1/2}$ by $[a^2 + \ell^2]^{-1/2} + [a^2 + \ell^2]^{-3/2}\ell x$ and $[a^2 + \ell^2 + 2\ell x]^{-1/2}$ by $[a^2 + \ell^2]^{-1/2} - [a^2 + \ell^2]^{-3/2}\ell x$, thus

obtaining $\ddot{x} = -GM[a^2 + \ell^2]^{-3/2}x$. In our study of oscillating motion in Chapter 10 (see (78) and (79) there), we found that if $\ddot{x} = -kx$, then the period of the motions is $2\pi/\sqrt{k}$. Using this result with $k = GM/[a^2 + \ell^2]^{3/2}$, we obtain the result of the text.

8. See Fig. 3. Regard the rod as consisting of particles of mass $(m/2\ell)dh$ at the distance h from the center of the wire. The attraction of the circular wire of mass M and radius a on this particle is by Exercise 4, $GM(m/2\ell)hdh/(a^2 + h^2)^{3/2}$. The attraction of the wire on the portion of the rod lying below is $(GM/2\ell)\int_0^{\ell-x} h\,dh/(a^2 + h^2)^{3/2}$. The attraction of the wire on the portion of the rod lying above is $(GMm/2\ell)\int_0^{\ell+x} h\,dh/(a^2 + h^2)^{3/2}$. The net attraction is the difference of these two integrals. To integrate let $a^2 + h^2 = u$. This gives the result in the text.

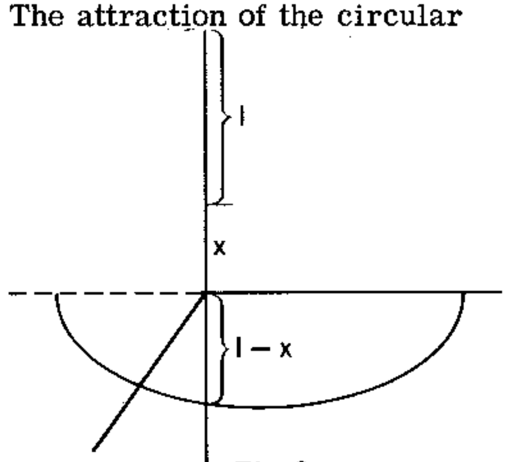


Fig. 3

9. The force acting on the rod at any position x is given by Exercise 8. This is precisely the force dealt with in Exercise 7. See the work above. Hence the rest of the argument is precisely that in Exercise 7.

CHAPTER 16, SECTION 6

1. The mass of the disc is $Mt(\pi a^2)$. The force between two points masses m_1 and m_2 separated by a distance r is $G\,m_1 m_2/r^2$. Hence we would have gotten $GMmt\pi a^2/h^2$. This does not agree with (41).
2. Use (41). Here $m = 2$, $h = 5$, $a = 10$. When one speaks of the mass per unit area of a thin disc, one usually is referring to the quantity Mt. Here $Mt = 3$. Computation of (41) gives the text's answer.
3. In (39) replace $\cos\theta$ by $h/\sqrt{x^2 + h^2}$ and $\sec^2\theta$ by $(x^2 + h^2)/h^2$ (see Fig. 16-15). We may now integrate by letting $u = x^2 + h^2$.
4. In (37) let $PS = r$. Replace $\cos\theta_i$ by h/r. Then (37) becomes $F = (kx_i h/r^3)\Delta x$ where $k = GMmt2\pi$. But $r^2 = x_i^2 + h^2$ so that $2rdr = 2x_i dx_i$. Then $F = (kh/r^2)\Delta r$. We may now integrate with respect to r. The limits of integration are h and $\sqrt{a^2+h^2}$. The result is again (41).
5. As a approaches ∞, F approaches $2\pi GMmt$. This is physically reasonable because if h is small most of the disc is pulling m at an unfavorable angle but if h is large, the opposite is true. Hence a result independent of h is reasonable.

6. The reasoning used in deriving (39) applies here, save that now we integrate from $x = 3$ to $x = 5$. Hence using the method of integration of the text we find $F = 2\pi GMmt\int_{\alpha_1}^{\alpha_2}\sin\theta\,d\theta = 2\pi GMmt(\cos\alpha_1 - \cos\alpha_2)$. Here α_1 is the value of θ corresponding to $x = 3$ and α_2 the value corresponding to $x = 5$ (see Fig. 16-15). Thus $\cos\alpha_1 = h/\sqrt{3^2 + h^2}$, $\cos\alpha_2 = h/\sqrt{5^2 + h^2}$. Since $h = 10$, we find $F = 20\pi GMmt(1/\sqrt{125} - 1/\sqrt{109})$.

7. If the mass m is at great distance from the disc, the disc may be thought of as a point mass. Thus by Exercise 1, the result should reduce to $\pi GmMta^2/h^2$. Now to confirm this by using (41), note that if we divide numerator and denominator in $h/\sqrt{a^2 + h^2}$ by h we have $1/\sqrt{1 + (a^2/h^2)} = (1 + a^2/h^2)^{-1/2}$. By binomial theorem, this last equals $1 - \frac{1}{2}(a^2/h^2)$ approximately. If we substitute this in (41) we confirm our physical argument. Of course if h becomes infinite the force becomes 0.

8. Following the suggestion of the text, we see that the force F_i on the i-th disc is obtained from (40) by replacing t by dh. Hence $F = 2\pi GmM(1 - \cos\alpha)\cdot\int_0^h dh$ and so we get the text's answer.

9. Following the suggestion of the text and replacing t in (41) by dh, we obtain $F = 2\pi GMm\int_c^{c+\ell}(1 - h/\sqrt{a^2 + h^2})dh$. By integration $F = 2\pi GMm\cdot\{\ell - \sqrt{(\ell + c)^2 + a^2} + \sqrt{c^2 + a^2}\}$.

10. In the derivation of (39) replace Mt by λx to obtain: $F \doteq (2\pi G\lambda h/h^2)\int_0^a (x^2\cos\theta/\sec^2\theta)dx$. Using the technique of the text, we obtain $F = 2\pi G\lambda h\int_0^\alpha (\sin^2\theta/\cos\theta)d\theta$. Use $\sin^2\theta = 1 - \cos^2\theta$ to obtain $F = 2\pi G\lambda h[-\sin\alpha + \log|\sec\alpha + \tan\alpha|]$, using integral formula #9. The values $\sin\alpha = a/\sqrt{a^2 + h^2}$, $\sec\alpha = \sqrt{a^2 + h^2}/h$, and $\tan\alpha = a/h$ yield the result given in the text.

CHAPTER 16, SECTION 7, FIRST SET

1. Following the suggestion, $F = \int_{D-a}^{\sqrt{D^2+a^2}}(GmMt\pi a/D^2)[1 + (D^2 - a^2)/r^2]\,dr$ $=(GmMt\pi a/D^2)\{[\sqrt{D^2 + a^2} - (D - a)] - (D^2 - a^2)[1/\sqrt{D^2 + a^2} - 1/(D - a)]\}$.
This can be simplified to $(2a^2/\sqrt{D^2 + a^2}) + 2a$.

2. We refer to Fig. 16-25. All the elements of a zone are at the same distance from 0, where m is located. The volume of the zone (see p.) is $t2\pi a^2\sin\theta\cdot d\theta$ where θ is the angle formed at 0 by 0Q and the horizontal from 0 to the right. The force of attraction which this zone exerts on m is $2GMm\pi a^2t\cdot\sin\theta\cos\theta\,d\theta/a^2$, the $\cos\theta$ entering because only the horizontal (in Fig. 16-25) component of the force acts. Now θ runs from 0 to $\pi/2$. Hence $F = 2GMm\pi t\cdot\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta = GMm\pi t$.

3. Replace t by dr in the result of Exercise 2. Then $F = \int_0^R \pi GMm\,dr = \pi GMmR$. Since $\overline{M} = 4\pi R^3 M/3$, this result may be also written as $3Gm\overline{M}/2R^2$.
4. The force between m and a particle of mass $\overline{M}$ at distance $h = R\sqrt{6}/3$ would be $Gm\overline{M}/h^2 = 3Gm\overline{M}/2R^2$, which agrees with the result of Exercise 3.
5. We need only to subtract the force due to a sphere of radius b from that due to a sphere of radius a. Thus in (54) we would integrate from b to a instead of from 0 to R.
6. By (53), the force due to a thin shell is $GMm4\pi r^2dr/D^2$. Now M = mass per unit volume $= \rho(r)$; hence $F = (GM/D^2)\int_0^R \rho(r)4\pi r^2dr$. The integral is the total mass, $\overline{M}$ say, of the sphere. Hence $F = G\overline{M}m/D^2$. That is, the sphere acts as though all its mass were D units from m.
7. We have that $G\overline{M}/R^2 = 32$ and $\overline{M} = 4\pi R^3\overline{\rho}/3$. Hence calculate $\overline{\rho}$.
8. Use the suggestion. Since, by the text, the attraction which the first sphere exerts on any small element of the second sphere is exactly the same as if the first sphere was concentrated at its center, we may replace the first sphere by a particle. But, then, using the text result again the attraction between a point and a sphere may be calculated by concentrating the mass of the sphere at its center. Hence the desired conclusion follows.

CHAPTER 16, SECTION 7, SECOND SET

1. By (58), $F = GMm4\pi\overline{h}/3$. Hence the work in moving the particle dh is $GMm4\pi h\,dh/3$. Thus $W = GMm(4\pi/3)\int_{R_1}^R h\,dh = GMm(4\pi/3)(R^2/2 - R_1^2/2)$.
2. From the text we know that only the part of the sphere "below" the particle acts on the mass. By Exercise 6 of the preceding section, we deduce that $F = (G/h^2)\int_0^h 4\pi\,\rho(r)r^2\,dr$ because $m = 1$ and the effective sphere has radius h.
3. Use the result of Exercise 2 with $\rho = R - r$. Then the attraction at h units from the center is $(G/h^2)\int_0^h 4\pi(Rr^2 - r^3)\,dr = 4\pi G(Rh/3 - h^2/4)$. As a function of h, this attraction is easily seen to be maximum at $h = \frac{2}{3}R$ from which the described result follows at once.
4. (a) Set up the usual division into sub-intervals. The heat in the i-th sub-interval is $c\Delta m\,T(x_i)$. Now $\Delta m = \rho(x_i)\,\Delta V$ where ΔV is an element of volume and $\Delta V = s\Delta x$. Thus $H = cs\int_0^a T(x)\rho(x)\,dx$.
   (b) $H = s\int_0^a c[T(x)]\,T(x)\rho(x)\,dx$.
   (c) $\Delta H = cT(r)\,\Delta m = cT(r)\cdot 4\pi r^2\rho(r)\,dr$; $H = 4\pi c\int_0^R r^2T(r)\,\rho(r)dr$.
5. Since the force on a point mass is $mr\omega^2$, $\Delta F = \Delta mr\omega^2$. Now $\Delta m = \rho\Delta V = \rho s\Delta r$; hence $F = \int_0^\ell \omega^2 rs\,\rho(r)\,dr$. We are given $\omega = 10$ rev./sec $= 20\pi$ rad/sec. Hence $F = 400\,\pi^2 s\int_0^\ell r\,\rho(r)dr$.

Solutions to Chapter 17

CHAPTER 17, SECTION 2

1. The half-line from 0 towards P in Fig. 17-4.
2. $\rho \sin\theta = 3$.
3. $\rho \sin\theta = -3$.
4. $\theta = 30° + n \cdot 180°$, $n = \pm 1, \pm 2, \ldots$.
5. (a) Circle, radius 5, center $(5, 0°)$.
   (b) Circle, radius 5, center $(5, 90°)$.
   (c) Circle, radius 5, center $(-5, 0°)$.
   (d) Circle, radius 5, center $(-5, 90°)$.
   (e) Straight line perpendicular to polar axis and 4 units to right of pole.
   (f) Two circles, each of radius $\frac{1}{2}$, centers $(\frac{1}{2}, 90°)$, $(-\frac{1}{2}, 90°)$.
   (g) Straight line perpendicular to polar axis and 4 units to left of pole.
   (h) Straight line parallel to polar axis and 1 unit above.

CHAPTER 17, SECTION 3

1. $\rho \cos\theta = -p$.
2. (a) an ellipse; (b) a parabola opening to the right;
   (c) an ellipse; (d) a hyperbola.
3. (a) When the directrix is perpendicular to and to the left of the pole formula (6) holds. Here $e = \frac{2}{3}$, $p = 6$; $\rho = 12/(3 - 2\cos\theta)$.
   (b) Use formula (8); $\rho = 12/(3 + 2\cos\theta)$.
4. Take a point P on the curve and draw $OP = \rho$ and the perpendicular from P to the directrix. Then $\rho/(p - \rho\sin\theta) = e$. Solve for ρ.
6. Ans. 8.
7. Same results as in Exercises 5 and 6.
8. From Exercise 5 we know that part of the major axis extends 6 units to the right of the pole and another part extends 2 units to the left. Hence the center is at $(2, 0)$.
9. The latus rectum is the width of the ellipse at the focus. The point on the ellipse above the focus is given by $\theta = \pi/2$. Hence half the latus rectum is ep.
10. As in Exercise 9, the point above the focus is given by $\theta = \pi/2$. Hence $\rho = p$ and the full latus rectum is 2p.
11. Use Fig. 17-11 as a guide. Part of the major axis is the ρ-value when $\theta = 0$. This ρ-value is $ep/(1 + e)$. The rest of the major axis is the ρ-value when $\theta = \pi$. This ρ value is $ep/(1 - e)$. The major axis is then $[ep/(1 + e)] + [ep/(1 - e)] = 2ep/(1 - e^2)$. Half of this is $ep/(1 - e^2)$. If we subtract this from $ep/(1 - e)$ we get $e^2p/(1 - e^2)$. The θ-value of the center is π or $-\pi$.

12. As in Exercise 11 we find ρ when $\theta = 0$ and when $\theta = \pi$. The sum of these two ρ-values is the major axis. Then take half.
13. We are given that $2ep = 2b^2/a$ and that $a = ep/(1 - e^2)$. Hence find b in terms of e and p.
14. The equation of any point on the curve, if the line to the directrix were the polar axis would be $\rho = ep/(1 + e\cos\theta)$. With the polar axis as in Fig. 17-15 all θ values of points on the curve are reduced by α, which means that $\theta + \alpha$ represents the same point that θ did.
15. If one chooses axes as in Fig. 17-16 then the focus-directrix definition of the ellipse calls for $PF/PD = e$ or $\sqrt{x^2 + y^2}/(x + p) = e$ and this gives $x^2(1 - e^2) + 2pxe + y^2 - p^2e = 0$. We can show this is an ellipse by using the $B^2 - 4AC$ test (See (40) of the Appendix to Chapter 7). Here $B = 0$, $A = (1 - e^2)$, $C = 1$. Hence $B^2 - 4AC = -4(1 - e^2)$. If $e < 1$, as it is for an ellipse, $B^2 = 4AC$ is negative.
16. Let the focus be the pole. Then the two parts of the chord are given by $r_1 = ep/(1 - \cos\theta)$ and $r_2 = ep[1 - \cos(\pi + \theta)]$. Then $1/r_1 + 1/r_2 = 2/ep$.

CHAPTER 17, SECTION 4

1. All of the parts in this Exercise are done by using the relations (14). The answers not in the text are: (b) $x = 5$; (d) $x^2 + y^2 + 4x = 0$; (f) $x^2 + y^2 = 25$; (g) First express $\sin 3\theta$ as $\sin(2\theta + \theta)$ and expand. $x^4 + 2x^2y^2 + y^4 - 15x^2y + y^3 = 0$; (i) $(x - 2)^2/16 + y^2/12 = 1$.
2. All of the parts in this Exercise are done by using (13). The answers not in the text are: (b) $\rho\cos\theta = 5$; (d) $\rho = 8\sin\theta$; (f) $\rho^2 = a^2\cos 2\theta$; (h) $\rho^2\sin^2\theta = 4\rho\cos\theta + 4$.
3. $\rho^2\cos^2\theta/a^2 + \rho^2\sin^2\theta/b^2 = 1$. This is not in the standard form of section 3 because the rectangular form presupposes that the center of the ellipse is the origin and this center is the pole of the polar coordinate system when one uses (13).

CHAPTER 17, SECTION 5, FIRST SET

1. Straightforward differentiation gives the answer.
2. (a) Find ρ' and set it equal to 0. From Exercise 1 we have $-e^2p\sin\theta = 0$. Then $\theta = 0$ and π. Now find ρ'' and one sees that for $\theta = 0$, ρ'' is negative; hence $\theta = 0$ furnishes a maximum. Similarly $\theta = \pi$ furnishes a minimum. The maximum and minimum values extend from the pole to the ends of the major axis.
3. ρ' is given by Exercise 1. It is negative for $0 < \theta < \pi$. Hence ρ decreases in this θ-interval.
4. Add the two ρ-values obtained in Exercise 2.

5. $\rho' = -p\sin\theta/(1-\cos\theta)^2$. Hence $\rho' = 0$ at $\theta = \pi$. ρ'' is positive at $\theta = \pi$. Hence a minimum. At $\theta = \pi$, $\rho = p/2$.
6. ρ' is negative for $0 < \theta < \pi$. Hence ρ is decreasing.
7. $d\rho/d\theta = 1/(d\theta/d\rho)$. The relationship is established by analytical means and so holds here too.

CHAPTER 17, SECTION 5, SECOND SET

1. In each case find ψ where $\tan\psi = \rho/\rho'$. The answers not in the text are (b) $\pi/6$; (d) $\pi/2$.
2. $\tan\psi = \rho/\rho' = -\cot\theta$. At $\theta = 30°$, $\tan\psi = -\sqrt{3}$. Since $\phi = \theta+\psi$, use (25). Then $\tan\phi = -\sqrt{3}/3$.
3. For $\rho = 1+\cos\theta$, $\rho' = -\sin\theta$. Hence $\tan\psi = -(1+\cos\theta)/\sin\theta$. At $\theta = \pi/3$, $\tan\psi = -\sqrt{3}$ and $\tan\theta = \sqrt{3}$. Now use (25). The answer is 0.
4. At $\theta = \pi/3$, $\tan\theta = \sqrt{3}$. $\tan\psi = \rho/\rho' = (a/\theta)/(-a/\theta^2) = -\theta$. At $\theta = \pi/3$, $\tan\psi = -\pi/3$. Now use (25). $\tan\phi = (3\sqrt{3}-\pi)/(3+\sqrt{3}\pi)$.
5. $\tan\psi = \rho/\rho' = 1/a$ and this is constant.
6. We start with $\tan\psi = \text{const.} = 1/a$, say. Then $\rho/\rho' = 1/a$ or $\rho' = a\rho$. Then $d\theta/d\rho = 1/a\rho$ and $\theta = (1/a)\log\rho + C$. Hence $\rho = c\,e^{a\theta}$ where $c = e^{aC}$.
7. The curve is a cardioid. $\tan\psi = \rho/\rho' = (1-\cos\theta)/\sin\theta$. At $\theta = \pi/2$, $\tan\psi = 1$. Hence $\psi = 45°$. Since $\phi = \theta + \psi$, $\phi = 135°$.
8. Denote the angle between OP and the tangent by α and the angle between PD and the tangent by β. Then we are to show $\alpha = \beta$. The angle ψ given by $\psi = \rho/\rho'$ is $\pi - \alpha$ since this formula yields the counterclockwise angle from OP to the tangent. Thus $\tan\alpha = \tan(\pi-\psi) = -\tan\psi = -\rho/\rho'$. If we extend the polar axis and the tangent, they meet at the angle β. Then $\theta = \alpha + \beta$. Hence $\tan\beta = \tan(\theta-\alpha) = (\tan\theta - \tan\alpha)/(1+\tan\theta\tan\alpha) = (\tan\theta + \rho/\rho')/[1 - (\rho/\rho')\tan\theta]$. We must show that $\tan\beta = \tan\alpha$. If we substitute the value of ρ/ρ' in the expressions for $\tan\rho$ and $\tan\alpha$ we find that they are equal.
9. Expect ψ to be $\pi/2$. If we calculate $\tan\psi$ at $\theta = 0$ we get $\tan\psi = \infty$ or $\psi = \pi/2$.
10. At a maximum or minimum $\rho' = 0$. Since $\tan\psi = \rho/\rho'$, $\tan\psi = \infty$ at a maximum or minimum. Hence $\psi = \pi/2$.
11. Use the left-hand figure in 17-20. Denote the length of the perpendicular from 0 to the tangent by d. Then $\tan\psi = d/\sqrt{\rho^2 - d^2}$. But $\tan\psi = \rho/\rho'$. Hence $d/\sqrt{\rho^2-d^2} = \rho/\rho'$. Solve for d.

CHAPTER 17, SECTION 6

1. $A = \frac{1}{2}\int_0^{2\pi} a^2(1-\sin\theta)^2\,d\theta = 3\pi a^2/2$.
2. $A = \frac{1}{2}\int_{-\pi/4}^{\pi/4} a\cos 2\theta\,d\theta = 2$.
3. $A = \frac{1}{2}\int_{\pi/2}^{\pi} \frac{1}{2}[64/(1-\cos\theta)^2]\,d\theta$. To integrate replace $(1-\cos\theta)^2$ by

$4\sin^4(\theta/2)$ and this by $\frac{1}{4}\csc^4(\theta/2)$. To integrate $\csc^4(\theta/2)$ write it as $[\cot^2(\theta/2)+1]\csc^2(\theta/2) = \cot^2(\theta/2)\csc^2(\theta/2) + \csc^2(\theta/2)$. Both of these terms are integrable because the derivative of $\cot x = -\csc^2 x$.

4. This is the same problem as Exercise 3 but using the Figure as in 17-11. The end values are 0 and $\pi/2$ instead of $\pi/2$ and π.

5. (a) $A = \frac{1}{2}\int_0^{2\pi} a^2\theta^2 d\theta = 4\pi^3 a^2/3$.

   (b) If we calculate $A = \frac{1}{2}\int_0^{4\pi} a^2\theta^2 d\theta$ we get $32\pi^3 a^2/3$. However the radius vector sweeps out the area between $\theta = 0$ and $\theta = 2\pi$ twice in two revolutions. Thus the answer is $28\pi^3 a^2/3$.

6. (a) $A = \frac{1}{2}\int_0^{2\pi} c^2 e^{2\theta} d\theta = (c^2/4)(e^{4\pi} - 1)$.

   (b) As in Exercise 5(b) if we integrate from 0 to 4π we cover the area from 0 to 2π twice. Hence we must subtract the result of integrating from 0 to 2π. Ans. $(c^2/4)(e^{8\pi} - e^{4\pi})$.

7. The circles intersect at $2\cos\theta = 1$ or $\theta = \pm(\pi/6)$. The required area is symmetric with respect to the polar axis. Hence we may consider the upper half. From $\theta = 0$ to $\theta = \pi/6$ it is bounded by $\rho = a$. From $\theta = \pi/6$ to $\theta = \pi/2$ it is bounded by $\rho = 2a\cos\theta$. Hence
   $A = \int_0^{\pi/6} a^2\, d\theta + \int_{\pi/6}^{\pi/2} 4a^2\cos^2 2\theta\, d\theta = (a^2/2)(5\pi/3 - \sqrt{3})$.

8. This false argument shows the danger of regarding an area as a sum of the infinite number of lines which the area might be supposed to consist of. The argument presupposes that each area is a sum of an infinite number of lines and just because each line of one is twice the corresponding line of the other, it concludes one area is twice the other. The argument is similar to Cavalieri's principle (as used in solid geometry) which is likewise not rigorous and so can lead to error.

CHAPTER 17, SECTION 7

1. (a) $s = \int_0^{2\pi} a\, d\theta = 2\pi a$.

   (b) $s = \int_0^{\pi} \sqrt{4a^2\sin^2\theta + 4a^2\cos^2\theta}\, d\theta = 2\pi a$.

   (c) $s = \int_{\pi/6}^{\pi} \sqrt{p^2/(1-\cos\theta)^2 + p^2\sin^2\theta/(1-\cos\theta)^4}\, d\theta = p\sqrt{2}\int_{\pi/6}^{\pi}(1-\cos\theta)^{-3/2}\, d\theta$.
   To integrate let $1 - \cos\theta = u$. Then $du = \sin\theta\, d\theta$ or $du/\sin\theta = du/\sqrt{u^2 - 2u}$.
   This change of variable leads to $p\sqrt{2}\int_{1-(\sqrt{3/2})}^{2} du/u^2\sqrt{u-1}$. Now use integral formula #27. Ans. $\sqrt[4]{3}/(2\sqrt{2} - \sqrt{6}) + (\sqrt{2}/4)\log|(\sqrt[4]{3} - 2)/(\sqrt[4]{3} + 2)|$.

(d) $s = \int_0^{2\pi} \sqrt{e^{2a\theta} + a^2e^{2a\theta}}\, d\theta = \int_0^{2\pi} \sqrt{1 + a^2}\; e^{a\theta}\, d\theta$. This gives the text's answer.

(e) $s = \int_0^{\pi} \sqrt{a^2\theta^4 + 4a^2\theta^2}\, d\theta = a \int_0^{\pi} \sqrt{\theta^2 + 4}\,\theta\, d\theta = [(\pi^2 + 4)^{3/2} - 8]/3.$

2 . $s = \int_0^{2\pi} \sqrt{a^2(1 - \cos\theta)^2 + a^2\sin^2\theta}\, d\theta = a\sqrt{2} \int_0^{2\pi} \sqrt{1 - \cos\theta}\, d\theta = 2a \int_0^{2\pi} \sin(\theta/2)\, d\theta$ $= 8a.$

3 . $s = \int_\alpha^\beta \sqrt{c^2e^{2a\theta} + a^2c^2e^{2a\theta}}\, d\theta = c\sqrt{1 + a^2} \int_\alpha^\beta e^{a\theta}\, d\theta = c\sqrt{1 + a^2}\,(e^{a\beta} - e^{a\alpha})/a$. But $\rho(\beta) - \rho(\alpha) = c(e^{a\beta} - e^{a\alpha})$.

4 . We have that $ds/d\theta = \sqrt{\rho^2 + \rho'^2}$. In Section 5, Second Set, Ex. 9, we proved that $d = \rho^2/\sqrt{\rho^2 + \rho'^2}$. Hence $ds/d\theta = \rho^2/d$.

5 . The complete curve is generated as θ goes from 0 to 3π. Then $s = \int_0^{3\pi} \sqrt{a^2\sin^6(\theta/3) + a^2\sin^4(\theta/3)\cos^2(\theta/3)}\, d\theta = \int_0^{3\pi} a\sin^2\theta\, d\theta = 3\pi a/2.$

CHAPTER 17, SECTION 8

1. We use the formula for the curvature K given immediately above in the text. By straightforward calculation we find the answers to (a), (b) and (c), namely,
   (a) $1/a$, (b) $2/a$, (c) $1/c\sqrt{1 + a^2}\, e^{a\theta}$.
   (d) In this part the calculation is rather extensive. We find that $\rho = p/(1 - \cos\theta)$, $\rho' = (-p\sin\theta)/(1 - \cos\theta)^2$, $\rho'' = (-p\cos\theta + p\sin^2\theta + p)/(1 - \cos\theta)^3$. We substitute these values in the expression for K and after considerable simplification obtain $K = (1 - \cos\theta)^{3/2}/2^{3/2}p$.
   (e) Straightforward calculation gives $K = (\theta^2 + 2)/a\sqrt{\theta^2 + 1}$.
   (f) Straightforward calculations gives $K = 3\rho/4$.
2. Using the value of K in 1(d) we have that $dK/d\theta = 3(1 - \cos\theta)^{1/2}\sin\theta/2^{5/2}p$. Then $dK/d\theta = 0$ for $\theta = 0$ and $\theta = \pi$. At $\theta = 0$, ρ is infinite. Hence this value of θ does not belong to a point on the curve. At $\theta = \pi$, $dK/d\theta$ changes from + to −. Hence $dK/d\theta$ is a maximum at $\theta = \pi$ and as Fig. 17-10 shows $\theta = \pi$ is the location of the vertex.
3. $R = c\, e^{a\theta}\sqrt{1 + a^2}$ and since $\rho = c\, e^{a\theta}$, $R = \sqrt{1 + a^2}\,\rho$.
4. Use of the formula for K and straightforward substitution leads to the text result. Near the end of the calculation replace $\sqrt{1 - \cos\theta}$ by $\sqrt{2}\sin(\theta/2)$.
5. The calculation of the curvature is lengthy but straightforward. We obtain $\rho = ep/(1 - e\cos\theta)$, $\rho' = -e^2p\sin\theta/(1 - e\cos\theta)^2$, $\rho'' = e^2p(e - \cos\theta + e\sin^2\theta)/(1 - \cos\theta)^3$, $K = (1 - e\cos\theta)^3/ep(1 - 2e\cos\theta + e^2)^{3/2}$. Now find $dK/d\theta$. This has the factor $\sin\theta$ which is 0 for $\theta = 0$. and $\theta = \pi$. The test that these furnish maxima is lengthy and perhaps not worth while carrying out.

Solutions to Chapter 18

CHAPTER 18, SECTION 2

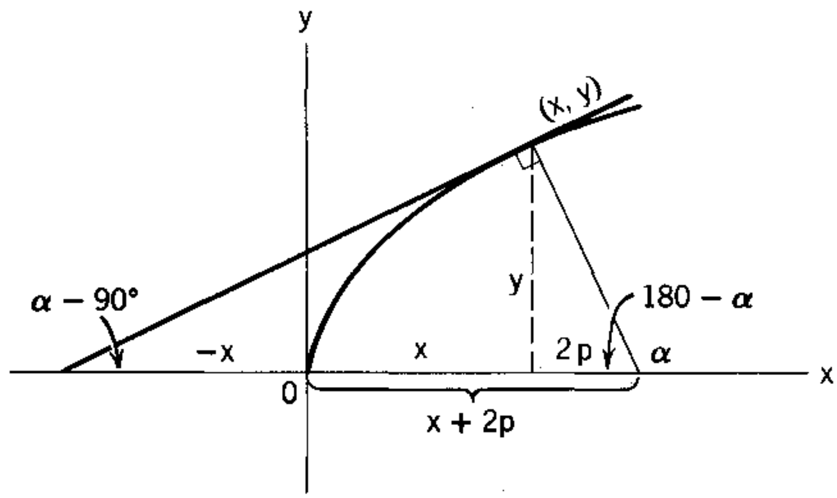
1. The direct equations for the parts not in the text are:
   (b) $y^2 = 4x$; (d) $y = x - 5\sqrt{x}$; (f) $4x^2 + y^2 = 4$;
   (h) $y = x$ for $-1 \le x \le 1$; (i) $y = x$ for $x > 0$.
2. (a) Using the arguments which led to (1) and (2) of the text we have $x = 50t$, $y = 16t^2$.
   (b) Let $t = 3$. Then $x = 150$, $y = 144$.
   (c) When the object hits the ground $y = 300$. Hence $t = 5\sqrt{3}/2$.
   (d) Calculate x when $t = 5\sqrt{3}/2$.
3. The horizontal motion is changed but not the vertical motion. Hence (c) remains the same.
4. Both the bullet and the apple have the same vertical motion, given by $y = 16t^2$. Hence when the bullet is x_1 ft from 0, its distance below OP will be that of the apple and the bullet will hit the apple. Note that the conclusion holds for any V.
5. If we add we get $x + y = 2$. This is what the graph of the parametric equations must yield.
6. Both objects have the same vertical motion, $y = 16t^2$. Hence they must reach the ground in the same time.
7. (a) Substitute $t = x/V$ in the equation for y.
   (b) The object will travel along the parabolic path until $y = 0$. Then $t = \sqrt{h}/4$. In this time $x = V\sqrt{h}/4$.
   (c) The range depends on V and $\sqrt{h}$. The range will increase more when V is increased than when h is.
8. We must first figure out how far from the target (horizontally) the bomb must be released. The bomb must fall 10,000 ft. and the vertical motion is given by $y = 16t^2$. Hence $t = 25$ when $y = 10,000$. In this time the bomb will travel 352·25 ft. Hence at the point at which the bomb is released, the target is 352·25 ft. away horizontally and 10,000 ft down. Then $\tan\theta = 352\cdot 25/10,000$ and $\theta = 41° 21'$.

CHAPTER 18, SECTION 3

1. Let t be the distance from the origin of any point on the line. Then $x = (\cos 45°)t$, $y = (\sin 45°)t$. Hence the answer in the text.
2. Let the distance of any point on the line from the origin be t. Then $x = t\cos\beta$ and $y = t\sin\beta$.
3. Yes. The motion would be faster.
4. $x = R\cos\theta$, $y = R\sin\theta$.

5. For (x, y) in the second quadrant, $x = -R\cos(90 - \theta)$, $y = R\sin(90 - \theta)$. Hence $x = -R\sin\theta$, $y = R\cos\theta$. If we consider (x, y) in other quadrants the results are the same.
6. In the second complete arc, the curve is exactly the same as in the first one. However, while the y-values remain the same, the x-values are larger by $2\pi R$. If we let θ increase from 2π on when the second arch is generated we see that x in (5) does increase by $2\pi R$ because of the $R\theta$ term, while the terms in $\sin\theta$ and $\cos\theta$ repeat their former values.
7. Replace θ in (5) by $2\pi t$.
8. Let Q be the foot of the perpendicular DP on the x-axis. It is obvious from triangle OQD that $x = a\cos\theta$. Since y equals the perpendicular CR from C onto the x-axis, we have from triangle ORC that $y = b\sin\theta$. Then $x = a\cos\theta$, $y = b\sin\theta$ are the parametric equations of P. But $x/a = \cos\theta$ and $y/b = \sin\theta$. Hence $(x/a)^2 + (y/b)^2 = 1$ and so (x, y) lies on the ellipse with semimajor axis a and semiminor axis b.
9. Let Q be the point where the radius drawn in Fig. 18-10 touches the circle. Note that since the string is taut $\angle OQP$ is a right angle and $PQ = a\theta$. Let B be the foot of the perpendicular from P to the x-axis and let R be the point in which the continuation of QP cuts the x-axis. Then $x = OB$, $y = PB$. We derive expressions for these lengths by using the two right triangles OQR and PBR and noting that $\angle QOR = \angle BPR = \theta$. Thus $x = OB = OR - BR$. From ΔOQR we have $OR = a\sec\theta$ and from ΔPBR, $BR = PR\sin\theta$. Thus $x = a\sec\theta - PR\sin\theta$. Finally from ΔOQR, $\tan\theta = (QP + PR)/a = (a\theta + PR)/a$. Solving this last relation for PR, we now obtain a value for BR. Then from $x = OR - BR$ we find $x = a(\cos\theta + \theta\sin\theta)$. From ΔPBR we have $y = PR\cos\theta$ and again using the relation for PR, we find $y = a(\sin\theta - \theta\cos\theta)$.
10. (a) Let Q be the foot of the perpendicular from P upon the x-axis. Draw the perpendicular from B to PQ and label its foot R. Then $\angle PBR = \theta$ and $\cos\theta = BR/BP = x/b$. Thus $x = b\cos\theta$. From ΔPAQ, we see that $\sin\theta = y/a$. Thus $y = a\sin\theta$.
    (b) Elimination of θ from $x = b\cos\theta$, $y = a\sin\theta$, as in Exercise 4, leads to the ellipse $(x^2/b^2) + (y^2/a^2) = 1$.
11. Let ϕ be the angle between PB and the x-axis (in the positive direction). Then $x = OB + PB\cos\phi$, $y = PB\sin\phi$. The triangle ABP is equilateral, so that $PB = a$. Clearly $OB = AB\cos\theta = a\cos\theta$. Finally since $\angle ABP = 60°$, we have $\phi = 180° - 60° - \theta = 120° - \theta$. Thus $x = a\cos\theta + a\cos(120° - \theta)$, $y = a\sin(120° - \theta)$.
12. (a) Note that $\angle NQO = \angle QOM = \theta$ and that $NQ = MP = y$. Thus from ΔNOQ, $\tan\theta = a/y$. Thus $y = a\cot\theta$. Now $x = OM = MP\cot\theta = y\cot\theta$. Thus $x = a\cot^2\theta$.

(b) Eliminating θ from $x = a\cot^2\theta$, $y = \cot\theta$, we find the parabola $x = ay^2$.



13. Using the results of Exercises 6 and 7 of Chapter 7, Section 4 we may label $x + 2p$ and $-x$ in the Figure. Then $\tan(180 - \alpha) = y/2p$ and $y = -2p\tan\alpha$. Further $\tan(\alpha - 90) = y/2x$. Hence, using the value of y, $x = p\tan^2\alpha$.
14. From the first equation we have $e^{x/c} = s/c + \sqrt{(s/c)^2 + 1}$. From the second we have $y^2/c^2 = s^2/c^2 + 1$ and $s/c = \sqrt{y^2/c^2 - 1}$. Hence $e^{x/c} = \sqrt{y^2/c^2 - 1} + y/c$. Transpose the last term, square both sides and solve for y. This gives $y = (c/2)(e^{x/c}+e^{-x/c})$.

CHAPTER 18, SECTION 4

1. Set $y = 0$ in (10) and solve for x. This gives (12).
2. It is shown on p.   that the maximum range is $V^2/32$. V in this Exercise is 1800 ft/sec.
3. From (12) we have $dx_1/dA = (V^2/16)\cos 2A$ and $d^2x_1/dA^2 = -(V^2/8)\sin 2A$. From $dx_1/dA = 0$ we have $A = 45°$ and we see that the second derivative is negative at $A = 45°$. Hence $A = 45°$ furnishes a maximum.
4. By (14) the maximum height when $A = 45°$ is $V^2/128$. This is $\frac{1}{4}$ of the maximum range which is $V^2/32$.
5. Since t_1 in (11) is 5, $V\sin A = 80$. From (12) where x_1 is now 450 we have $V^2 \sin 2A = 32 \cdot 450$. Now $\sin A = 80/V$ and $\cos A = \sqrt{V^2 - 6400}/V$. We substitute these values in $V^2\sin 2A = 32\cdot 450$ where $\sin 2A = 2\sin A\cos A$. Then $160\sqrt{V^2 - 6400} = 32 \cdot 450$ and $V = 120$ ft/sec. approx.
6. This is just a modification of Exercise 5. Here we find A instead of V. From (11) we have $V\sin A = 16T$ and from (12) we have $V^2\sin 2A = 32X$. From the first equation $V = 16T/\sin A$. Substitute this in the second and use $\sin 2A = 2\sin A\cos A$. This gives the text's result for A.
7. From (14) it is clear that the shell attains maximum height when $\sin A = 1$ or $A = 90°$.

8. (a) The expression for range, given by (12), is $x_1 = (V^2/32)\sin 2A$. Given a value of x_1 (which must be less than the maximum range of $V^2/32$) we have $\sin 2A = 32x_1/V^2$. Then there are two values of $2A$ between 0 and 180°. If one value is $2A_1$, the other must be $180 - 2A_1$. Then the two angles of fire are A_1 and $90 - A_1$.
   (b) By (11) we see that the smaller of A_1 and $90 - A_1$ gives the shorter time of flight.
9. Exercise 4 of Section A3 of Chapter 7 gives tne relevant information for $y = ax^2 + bx + c$. In our case $a = -16/V^2\cos^2 A$, $b = \tan A$, and $c = 0$. We have but to substitute these values of a, b and c into the results of that earlier Exercise to obtain the answers desired here. One could, of course, translate axes and put the equation of the projectile path into the form $y = (1/4p)x^2$ and so obtain the results independently.
10. We wish to eliminate A between equations (14) and (15). From (14) we have $\sin A = \sqrt{64y/V^2}$. Hence $\cos A = \sqrt{1 - 64y/V^2}$. Substitute these values in (15), since $\sin 2A = 2\sin A\cos A$. Simplifying and then completing the square gives $x^2/(V^2/64)^2 + (2y - V^2/64)^2/(V^2/64)^2 = 1$.
11. Since the initial horizontal velocity is now $V\cos A + 88$, equation (6) of the text must be replaced by $v_x = V\cos A + 88$. Thus in place of (7) we obtain $x = (V\cos A + 88)t$. By the principle of independence of motions, equation (9) is unchanged and so (11) is still correct. If we substitute the value of t from (11) into our new expression for x we obtain the text answer.
12. The time when the object strikes the ground is given by (11). If we substitute this value of t in (6) and (8) [Of course, (6) is independent of t] we obtain the components of the velocity when the shell strikes the ground. These are $v_x = V\cos A$ and $v_y = -V\sin A$. The magnitude of this velocity is V, the original magnitude. However the direction of resultant velocity, if taken to be the inclination of the tangent of the path at the point where the shell strikes the ground is $180 - A$. Alternatively the direction between the velocity vector and the positive x-axis is A.
13. (a) From (6) and (8) we have that $\sqrt{v_x^2 + v_y^2}$ gives the text's answer.
    (b) From the text's answer to (a) we obtain $\sqrt{V^2 - 64Vt\sin A + (32t)^2}$. Now eliminate $Vt\sin A$ by using its value in (9). Then we get the text's answer.

14. Use the result of 13(a) to find dv/dt, where v is the magnitude. Set $dv/dt = 0$ and we find that $t = V \sin A/32$. But by (13) this value of t is the time at which the projectile is at the maximum height.

15. $\operatorname{Tan} B = y/x$. Use the values of y and x given by (9) to affirm $\tan B = (-16t^2 + Vt \sin A)/Vt \cos A$. Now solve for V.

16. By the principle of independence of motions, the horizontal motion is unchanged from the derivation of the text. Thus by (7) we have $x = Vt \cos A$. If we measure y from the ground then in the equation leading to (9), we evaluate C by the conditions $y = h$ at $t = 0$. Thus in place of (9) we have $y = 16t^2 + Vt \sin A + h$. Using this new value of y we find t when $y = 0$. The positive value of t is $(V \sin A + \sqrt{V^2 \sin^2 A + 64h})/32$. If we substitute this in (6) and merely use $\sin 2A = 2 \sin A \cos A$ we get the text's answer.

17. The range is given by the result of Exercise 16. We are to maximize r with respect to A. Hence neglecting the constant factor $V^2/64$, and using K for $256h/V^2$ we find dr/dA and set it equal to 0. This gives $4 \cos 2A \cdot \sqrt{\sin^2 2A + K \cos^2 A} = K \sin 2A - 4 \sin 2A \cos 2A$. Squaring and cancelling $16 \sin^2 2A \cos^2 2A$, allows division by $4K \cos^2 A$. This gives $4 \cos^2 2A = K \sin^2 A - 8 \sin^2 A \cos 2A$. Replace $\cos 2A$ by $2 \cos^2 A - 1$ and $\sin^2 A$ by $1 - \cos^2 A$ and simplifying gives $4 = K - K \cos^2 A + 8 - 8 \cos^2 A$. Solving for $\cos^2 A$ gives the text result. We could calculate d^2r/dA^2 and show it is negative at the value of A just found.

Suppose that the angle A of Figure 18-15, which gives maximum range, were $\geq 45°$. Trace the corresponding parabola backward to the ground level and suppose it cuts the ground at some point P. It is geometrically obvious that the angle of "fire" at P, angle B say, is greater than $45°$. Consider the trajectory starting at P and making an angle of $45°$ with the ground at P. This trajectory has greater range than the one belonging to angle B. However this $45°$ trajectory will cut the line segment h at some point M, say, which is below the point, L say, at which the ball is hit. Thus we have a trajectory launched at M which has a greater range than the one launched at L, despite the fact that M is below L. This is contrary to intuition.

Since L is above M we should be able to find a trajectory emanating from L which has a greater range than the one from M. Hence A must be less than 45°.

This argument shows something additional. The trajectory launched at L with maximum range should when traced backward to P, say, make an angle of 45° with the ground. For if this latter angle were more than 45° we could take the trajectory with a 45° angle at P and use the argument of the preceding paragraph to arrive at a "contradiction". If the angle at P were less than 45° we could take that trajectory through L which does make a 45° angle with the ground and hits it at some point S, necessarily to the right of P. This latter trajectory would have greatest range of all through S and since S is to the right of P, more range from L than the one starting at P.

When $h = 0$, $K = 0$ and $A = \cos^{-1}\sqrt{1/2}$. Hence $A = 45°$.

18. From Exercise 17 we have $\cos A = \sqrt{(K+4)/(K+8)}$. Hence $\sin A = \sqrt{4/(K+8)}$ and since $\sin 2A = 2\sin A\cos A$ we have the necessary value to calculate the maximum range r by substituting these values of $\sin A$ and $\cos A$ in the result of Exercise 16. The maximum range is given in the text.

19. As noted in the suggestion, when the projectile strikes the line OQ the values $x = r\cos B$, $y = r\sin B$ satisfy the equation (10) of the projectile's path. Inserting these values into (10) and rejecting the root $r = 0$ (the origin), we find that $r = (V^2/16)\cos A\sin(A - B)/\cos^2 B$. The answer in the text can readily be reduced to the one just given by expanding $\sin(2A - B)$ and replacing $\cos^2 A$ by $2\cos^2 A - 1$. The text answer is better for the next Exercise.

20. We use the result of Exercise 19 to maximize r with respect to A. Since B is constant we find from $dr/dA = 0$ that $\cos(2A - B) = 0$ or $2A - B = 90°$. Let $A = (90 + B)/2$ be inserted in the expression for r. We get $r = (V^2/32)\cdot(1 - \sin B)/\cos^2 B$. Since $\cos^2 B = 1 - \sin^2 B$ we factor and get the text answer.

21. Write $r = V^2/32(1 + \sin B)$ in the notation $\rho = (V^2/32)(1 + \sin\theta)$ and note formula (9) of Chapter 17. Since $e = 1$ in our case the locus is a parabola.

22. (a) Firing the gun at angle $A - B$ to the vertical means firing at an angle $90° - A + B$ to the horizontal. The range for any firing angle A is given by the result of Exercise 19. Replacing A by $90° - A + B$ in that formula leaves it unchanged. This is just the desired conclusion.

(b) The bisector in question makes an angle of $45° + B/2$ with the horizontal. It follows that both the angles A and $90° - A + B$ make an angle of $45° + B/2 - A$ with this bisector.

23. The result of Exercise 20 with the values $V = 600$ ft/sec and B determined by $\tan B = 1/20$ leads to the answer 3550 yds (approx.).

24. As a consequence of (12) it was shown that the maximum range is $V^2/32$. Thus the horizontal area endangered is a circle with this radius. The area is $\pi(V^2/32)^2$.

25. We are given the maximum height that the stone can attain, namely 10 ft. However the maximum range may not occur for a path with maximum height of 10. We shall rather take the value of $\sin A$ from (14), with $V = 80$, namely $\sin A = \sqrt{y_2/100}$ and substitute it in (12), which gives the range for any A. Then, since $\sin 2A = 2 \sin A \sqrt{1 - \sin^2 A}$, $x_1 = 200 \cdot 2\sqrt{y_2/100} \cdot \sqrt{1 - y_2/100} = 4\sqrt{100y_2 - y_2^2}$. We now maximize x_1 with respect to y_2. We can as well maximize $q = 100y_2 - y_2^2$. Now $dq/dy_2 = 100 - 2y_2$. This is positive for $y_2 < 50$ so that q is an increasing function. Hence q and therefore x_1 attains its maximum when $y_2 = 10$. Substituting this value of y_2 in our expression for x_1 we obtain $x_1 = 120$.

To find the time of flight we use (11). However we need $\sin A$. But this is $\sqrt{y_2/100} = \sqrt{1/10}$. Hence $t_1 = {}^{80}/_{16}\ \sqrt{0.1} = 5\sqrt{0.1} = \sqrt{10}/2$.

We could save work if we concluded at once on the basis of intuition that 10 feet is the height of the path which gives maximum range.

26. From (10) with $V = 160$ and $x = 480$, we find the height y is given in terms of the firing angle A by $y = -144 \sec^2 A + 480 \tan A$. Setting $dy/dA = 0$ and noting that $\sec^2 A \neq 0$, we obtain the condition $\tan A = {}^5/_3$. Thus $A = 59°$ (approx.).

27. Use formula (12) with $A = 30°$, $V = 120$ to obtain the range which is $225\sqrt{3}$ ft.

28. We use (10) in which x and y are now fixed to find A. Since $\sec^2 A = 1 + \tan^2 A$ we obtain a quadratic in $\tan A$ and we find that $\tan A = V^2/32 \pm \sqrt{(V^2/32)^2 - 2(V^2/32)y - x^2}$. The roots are complex, real and distinct, or real and equal according as the discriminant is negative, positive or 0. If the roots are complex, there is no value of A for which the point (x, y) can be reached. If the roots are real and distinct there are two value of A. If the roots are equal there is one value of A.

(b) The condition that the roots be equal, namely, $(V^2/g)^2 - 2(V^2/g)y - x^2 = 0$, is itself the equation of a parabola. This is all that students can be expected to state at this point. The teacher might care to point out that this parabola is the envelope of the family of parabolas which one gets by keeping V fixed and varying A in equation (10), that is, the family of projectile paths for different A-values. The theory of envelopes is treated in Chapter 22, section 9 and in fact the problem of finding the envelope of this family of projectile paths is Exercise 9 of that section. The envelope is also the parabola of surety mentioned in Exercise 21 above.

29. Use the result of Exercise 16 in which we put $h = 700$, $A = 45°$, and $V = 100$. The arithmetic yields 31,900 ft. approx.

30. From (7) and (9) we have $x = Vt \cos A$, $y = -16t^2 + Vt \sin A$. For fixed V and t these are the parametric equations of the locus of the particles at that instant. Thus if we eliminate A between these equations, we shall have the direct equation of the (x, y) values for fixed V and t. We find $x^2 + (y + 16t^2)^2 = V^2t^2$. For fixed V and t this is a circle with center $(0, -16t^2)$ and radius Vt.

31. (a) In Exercise 9(a) we found the directrix of the parabolas given by equation (10). The directrix is independent of A so that all the parabolas have the same directrix. Ans. $y = V^2/64$.

(b) From (14) we obtain the height when $A = 90°$. This is $V^2/64$ and this is the height of the directrix.

(c) From Exercise 9(c) the focus of any one parabola has the coordinates $x = V^2 (\sin 2A)/64$, $y = -V^2 (\cos 2A)/64$. These two equations are the parametric equations of the locus of the foci, with A as the parameter. If we eliminate A (by solving for $\sin 2A$ and $\cos 2A$ respectively) we obtain $(64x^2/V^2)^2 + (64y^2/V^2)^2 = 1$. This locus is a circle with center at the origin, which is where the gun is.

(d) The level of the focus is given by its y-value, namely, $y = -V^2(\cos 2A)/64$. When $A = 45°$, $y = 0$. When $A > 45°$, y is positive and when $A < 45°$, y is negative. The gun is at $y = 0$ (and $x = 0$).

(e) The coordinates of the vertex of any one parabola are given in Exercise 9(b). These are $x = V^2 (\sin 2A)/64$ and $y = V^2 (\sin^2 A)/64$. These two equations are the parametric equations of the locus of the vertices, with A as the parameter. To get the direct equation we must eliminate A. From the second equation we have $\sin A = \sqrt{64y/V^2}$. If we substitute this in $x = V^2 2 \sin A \sqrt{1 - \sin^2 A}/64$ and simplify we obtain $(32)^2x^2 + (64)^2y^2 - 64V^2y = 0$. We can, by completing the square, rewrite this as $x^2/(V^2/64)^2 + (y - V^2/128)^2/(V^2/128)^2 = 1$. This is the equation of an ellipse with center at $(0, V^2/128)$. This center is halfway between the gun at $(0, 0)$ and the directrix at $V^2/64$.

32. (a) Taking the position of the gunner as the origin, the trajectory of the bomb is $x = x_1$, $y = -16t^2 + h$. The trajectory of the gunner's shell is $x = Vt \cos A$, $y = y = -16t^2 + Vt \sin A$. Since these trajectories are to intersect, we have $x_1 = Vt \cos A$, $h = Vt \sin A$. Thus $\tan A = h/x_1$. However, h/x_1 is just the tangent of the angle of sight from the gunner's position to the point where the bomb begins to fall. Thus the desired conclusion holds.

(b) The argument in (a) is to the effect that by aiming at the point where the bomb begins to fall the gunner will hit the bomb. No special value of the velocity had to be specified. Hence the gunner will hit the bomb no matter what the velocity (unless it is so small that the bomb reaches the ground before being hit). The place at which the bomb is hit, that is, the distance it falls before being hit, will vary with the velocity of the shell fired by the gunner. But the bomb will be hit.

33. This problem is really the same as Exercise 24. The various angles at which the soldier can throw the grenade are the same as the angle at which the fragments of shrapnel disperse. Hence the answer is the same.

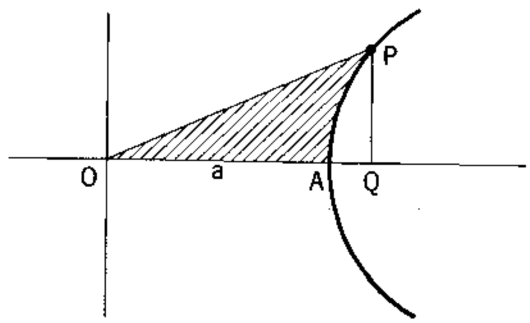
CHAPTER 18, SECTION 5, FIRST SET

1. We use (18) to obtain dy/dx and (24) to obtain d^2y/dx^2.
   (b) $dy/dx = 2/3t$, $d^2y/dx^2 = -1/9t^3$.
2. The slope at any instant is $\dot{y}/\dot{x} = (-32t + V \sin A)/V \cos A$. When the shell strikes the ground, $y = 0$ and $t = V \sin A/16$. Substitute this value of t in $\dot{y}/\dot{x}$ and we get $-\tan A$.
3. Since $dy/dx = \dot{y}/\dot{x}$, $dy/dx = 0$ when $\dot{y} = 0$. Using the value of y we have $\dot{y} = 0$ when $t = V \sin A/32$. We can now use (24) to see that d^2y/dx^2 is negative at this value of t. To obtain the maximum y-value we substitute this value of t in the expression for y and we get the text's answer.
4. Since $dy/dx = \dot{y}/\dot{x}$, $dy/dx = -b \cot\theta/a$. Also $d^2y/dx^2 = -b \csc^3\theta/a^2$. $dy/dx = 0$ at $\theta = \pi/2$ and $\theta = 3\pi/2$. d^2y/dx^2 is negative at $\theta = \pi/2$ and positive at $\theta = 3\pi/2$. Since $y = b \sin\theta$, we have $y = b$ at the maximum and $y = -b$ at the minimum.
5. Using $\dot{y}/\dot{x}$ we get $dy/dx = \sin\theta/(1 - \cos\theta) = \sqrt{1 - \cos^2\theta}/(1 - \cos\theta)$ $= \sqrt{(1 + \cos\theta)/(1 - \cos\theta)} = (\cos\theta/2)/(\sin\theta/2)$.
6. Since $dy/dx = \cot(\theta/2)$, $dy/dx = 0$ when $\theta = \pi$. $d^2y/dx^2 = -\csc^2(\theta/2)/2$ and this is negative when $\theta = \pi$. When $\theta = \pi$, $y = 2R$.
7. The normal to a curve $y = f(x)$ through a point (x_0, y_0) on the curve is given by $y - y_0 = [-1/f'(x_0)]/(x - x_0)$. From Exercise 5, $f'(x_0) = \cot(\theta/2)$ where θ is the value of the parameter corresponding to $x = x_0$, $y = y_0$. Thus using (5), the normal is $y - R(1 - \cos\theta) = -\tan(\theta/2)[x - R(\theta - \sin\theta)]$. In the discussion leading to (5) it was pointed out that T has the coordinates $x = R\theta$, $y = 0$. Thus we need only verify that this point lies on the normal. That is, we must verify that the equation $-R(1 - \cos\theta) = -\tan(\theta/2)(R \sin\theta)$ holds. This condition simplifies to $\tan(\theta/2) = (1 - \cos\theta)/\sin\theta$, which holds since it is one of the half angle formulas for trigonometry. (See Exercise 5).
8. Proceed as in Exercise 7, using the equation $y - y_0 = f'(x_0)(x - x_0)$ of the tangent and the fact that the point diametrically opposite T has coordinates $x = R\theta$, $y = 2R$.
9. From the second equation in (5), $\cos\theta = 1 - (y/R)$ so that $\theta = \cos^{-1}[(R - y)/R]$ and $\sin\theta = \pm\sqrt{1 - \cos^2\theta} = \pm\sqrt{y(2R - y)}/R$. Substituting these values in the parametric equation for x we get the text's answer.
10. Use (5) and (24). The calculation is straightforward. The curve is always concave downward.
11. From $x = \rho\cos\theta$ where ρ is a function of θ, $\dot{x} = -\rho\sin\theta + \cos\theta\, d\rho/d\theta$. Likewise $\dot{y} = \rho\cos\theta + \sin\theta\, dp/d\theta$. Hence by (18) the text result holds.
12. From the result in Exercise 9 we may calculate dx/dy and then use the fact that $dy/dx = 1/(dx/dy)$. $dx/dy = [1 \pm (R - y)]/\sqrt{2Ry - y^2}$.

CHAPTER 18, SECTION 5, SECOND SET

1. The integration is straightforward.
2. By (29) we have $A = \int_0^{2\pi} b\sin\theta\,(-a\sin\theta)d\theta = -\pi ab$. The geometrical area is πab.
3. (a) Since $\cosh^2 u - \sinh^2 u = 1$, we see that $x^2/a^2 - y^2/b^2 = 1$. This is the equation of a hyperbola. Since $\cosh u$ is positive the original equations represent only the right hand branch of the hyperbola.

   (b) In the Figure P has the coordinates $(a\cosh u_0, b\sinh u_0)$; thus Q is $(a\cosh u_0, 0)$. The desired area (shown shaded) is the area of the



   triangle OPQ minus the area under the arc of the hyperbola from A to P. Thus $A = \frac{1}{2}\, a\cosh u_0 \cdot b\sinh u_0 - \int_0^{u_0} b\sinh u \cdot a\sinh u\,du$. Formula 109 of the integral tables and the relation $\sinh 2u = 2\sinh u\cosh u$ enable us to obtain the result $A = ab\,u_0/2$.
4. The entire curve is described as θ goes from 0 to 2π. $A = \int_0^{2\pi} y(\theta)\dot{x}(\theta)d\theta$ $= \int_0^{2\pi} 2\sin^3\theta\cdot 6\cos^2\theta(-\sin\theta)\,d\theta = \int_0^{2\pi} -12\sin^4\theta\cos^2\theta\,d\theta$. We can save some work here. The area is $4\int_0^{\pi/2} -12\sin^4\theta\cos^2 d\theta$. Now use formula 118 of the Integral Table with $m = 4$ and $n = 2$.

CHAPTER 18, SECTION 5, THIRD SET

1. By (34), $s = \int_0^{2\pi}\sqrt{a^2\sin^2\theta + a^2\cos^2\theta}\,d\theta = a\int_0^{2\pi} d\theta = 2\pi a$.
2. In this Exercise we use formula (34) wherein t_2 is now t and $t_1 = 0$.
   (b) $s = 3t$; (d) $s = 25t^2$.
3. As in Exercise 2, we have $s = \int_0^5 \sqrt{(2t)^2 + (2)^2}\,dt = 2\int_0^5 \sqrt{t^2 + 1}\,dt.$ Using formula 40 of the integral tables, we find $s = 5\sqrt{26} + \log(5 + \sqrt{26})$.

4. Following the suggestion we have $x = (\cos\theta + \theta\sin\theta)$, $y = (\sin\theta - \theta\cos\theta)$. Thus $s = \int_0^{6\pi} \sqrt{(-\sin\theta + \sin\theta + \theta\cos\theta)^2 + (\cos\theta - \cos\theta + \theta\sin\theta)^2}\, d\theta$. This simplifies to $s = \int_0^{6\pi} \theta\, d\theta$. Hence $s = \frac{1}{2}(6\pi)^2 = 177.7$ in (approx.)
5. If we attempt to use (38) except for the change to the end values π and 3π instead of 0 and 2π we obtain 0. The incorrect answer results from the fact that the quantity $\sin(\theta/2)$ in (38), which is the essential part of $ds/d\theta$, is negative when θ runs from 2π to 3π, though it is positive when θ runs from π to 2π. Hence one must evaluate (38) first from π to 2π and then from 2π to 3π. The latter gives $-4R$ and the sign must be ignored if we are interested in the actual arc length.
6. (a) By (34), $s = \int_0^2 t\sqrt{9t^2 + 4}\, dt$. To integrate let $u = 9t^2 + 4$. We find $s = (40^{3/2} - 4^{3/2})/27$.

   (b) Since $\dot{x}$ is negative for $t < 0$, we must write (cf. see the footnote in the text) $s = -\int_{-2}^{0} t\sqrt{9t^2 + 4}\, dt + \int_0^2 t\sqrt{9t^2 + 4}\, dt$. Making the substitution $t = -\theta$ in the first integral, we find $s = 2\int_0^2 t\sqrt{9t^2 + 4}\, dt = \frac{2}{27}(40^{3/2} - 4^{3/2})$.

CHAPTER 18, SECTION 5, FOURTH SET

All of the Exercises in this list call for no more than straightforward calculations using (40). The answers not in the text are:

2. (b) $5\sqrt{5}/2$; (d) $15\sqrt{5}$; (e) $-1/\sqrt[3]{2}\sqrt{1 - \cos 20t}$.
4. $1/4R$.
5. The vertex is given by $u = 0$.

CHAPTER 18, SECTION 6

1. In Chapter 10, we found that the solution of $\ddot{x} = -kx$ is $x = A\cos\sqrt{k}\, x + B\sin\sqrt{k}\, x$. Here $k = 32/4R = 8/R$. Hence $s = A\cos\sqrt{8/R}\, t + B\sin\sqrt{8/R}\, t$. At $t = 0$, we have $s = s_0$, $\dot{s} = 0$. Thus we find $s = s_0\cos\sqrt{8/R}\, t$.
2. From (49) the period is $\pi\sqrt{R/2}$ no matter what the initial amplitude is.
3. Since the particle slides under gravity, it moves just like the bob of the cycloidal pendulum.
4. By virtue of the fact that the slide from D to B is merely the reverse in the acceleration and velocity involved, the time from A to D is (see (49) $\pi\sqrt{R/32}$. But the period is independent of the amplitude and so this is also the time to go from (0, 2R) to D.

5. At D, where $\theta = \pi$, we have that $x = \pi R$, $y = 0$. Then the line joining $(0, 2R)$ and $(\pi R, 0)$ has the slope $2R/-\pi R = -2/\pi$. This is $\tan\alpha$ and so $\sin\alpha = 2/\sqrt{4 + \pi^2}$. Now we treat the motion down the line as the motion along an inclined plane with inclination α (or $180 - \alpha$). The acceleration down the plane is $32 \sin\alpha$. Hence $\ddot{s} = 32 \sin\alpha$, $\dot{s} = 32t \sin\alpha + C$. Since $\dot{s} = 0$ when $t = 0$, then $C = 0$. Then $s = 16t^2 \sin\alpha$, because the constant of integration is 0 if we measure s from $(0, 2R)$. Now we wish to calculate the time to travel the distance s from $(0, 2R)$ to $(\pi R, 0)$. This distance is $\sqrt{\pi^2 R^2 + 4R^2} = R\sqrt{4 + \pi^2}$. Substitute this value of s and the value of $\sin\alpha$ in $s = 16t^2 \sin\alpha$ and solve for t. This gives $t = \sqrt{R(4 + \pi^2)/32}$. This time is clearly more than $\pi\sqrt{R/32}$ which holds for the cycloidal path.

CHAPTER 18, SECTION 7, FIRST SET

1. (a) $3\vec{i}+4\vec{j}$; (b) $2\vec{i}+3\vec{j})$; (c) $-2\vec{i}-3\vec{j}$; (d) $(\sqrt{2}/2)\vec{i}+(\sqrt{2}/2)\vec{j}$; (e) $(\sqrt{3}/2)\vec{i}+(1/2)\vec{j}$; (f) $(1/\sqrt{5})\vec{i}+(2/\sqrt{5})\vec{j}$; (g) $(2\vec{i}+3\vec{j})+(4\vec{i}+4\vec{j}) = 6\vec{i}+7\vec{j}$; (h) The slope of $y = x^2$ at $x = 2$ is 4. A vector having this slope is $\vec{i}+4\vec{j}$. However, to make it a unit vector divide each component by $\sqrt{17}$.
2. The magnitude of $a\vec{i}+b\vec{j}$ is $\sqrt{a^2+b^2}$.
3. If the initial point of the vector $\vec{v}$ is at the origin then $x = \cos\theta$ and $y = \sin\theta$ so that $x^2+y^2 = 1$.
4. Same method as 1(h). $\vec{i}+6\vec{j}$ with the components divided by $\sqrt{37}$ to make a unit vector. We could also include the oppositely sensed vector by dividing by $-\sqrt{37}$.

CHAPTER 18, SECTION 7, SECOND SET

1. The velocity is given by $d\vec{r}/dt$ and the acceleration by $d^2\vec{r}/dt^2$.
2. $\vec{v} = -\omega R \sin\omega t\vec{i}+\omega R\cos\omega t\vec{j}$; $\vec{a} = -\omega^2 r\cos\omega t\vec{i}-\omega^2 r\sin\omega t\vec{j}$. The magnitudes are the square root of the sum of the square of the components.
3. $\vec{r}(t) = r\cosh\omega t\vec{i}+r\sinh\omega t\vec{j}$. Find $d\vec{r}/dt$ and $d^2\vec{r}/dt^2$.
4. (a) The parametric equations of the cycloid (see (5) and Fig. 18-7) are $x = R(\theta-\sin\theta)$, $y = R(1-\cos\theta)$ or the vector equation is $\vec{r}(\theta) = R(\theta-\sin\theta)\vec{i}+R(1-\cos\theta)\vec{j}$. Let $\theta = \omega t$. We can now calculate the vector velocity $d\vec{r}/dt$ and the vector acceleration $d^2\vec{r}/dt^2$. Then we obtain for the magnitude of the velocity $2R\omega\sin(\omega t/2)$. We see that the magnitude of the velocity is zero for $\omega t/2 = 0$ or π, that is for $t = 0$ or $2\pi/\omega$. At these times the particle is at its lowest point.

(b) $|v|$ is maximum for $\sin(\omega t/2) = 1$, that is, for $\omega t/2 = \pi/2$ or for $t = \pi/\omega$. Since $\theta = \omega t$, at $t = \pi/\omega$, $\theta = \pi$. If we examine Fig. 18-7 we see that when $\theta = \pi$, P is at the top of its path. At that point the point P moves along the cycloid with two velocities. P moves on the circle with a linear velocity of $R\omega$, since ω is the angular velocity of the circle. Also, the center C of the circle is constantly moving to the right. Its velocity is the distance it covers in a full revolution divided by the time it takes to make a full revolution or $2\pi R/(2\pi/\omega) = R\omega$. Though both the rotational motion of P and the translational motion of C contribute to the motion of P along the cycloid these two motions are generally not in the same direction and do not add except when P is at the highest point and then the directions are the same. Hence then the linear velocity of P is $R\omega + R\omega$. If we examine $|\vec{v}|$ we see that when $t = \pi/\omega$, $|\vec{v}|$ is $2R\omega$.

5. The information needed to make the sketch is given by $d\vec{r}/dt$ and $d^2\vec{r}/dt^2$. $\dot{x}$ and $\dot{y}$ are the components of the velocity vector and $\ddot{x}$ and $\ddot{y}$ are the components of the acceleration vector.

6. We have that $ds/dt = 2R\omega \sin(\omega t/2)$. Then $s = 2R\omega \int_0^t \sin(\omega t/2)dt$. This gives the text answer.

7. For a complete arc, $\theta = 2\pi$ or ωt goes from 0 to 2π. Hence $t = 2\pi/\omega$. Then the result of Exercise 6 gives $8R$.

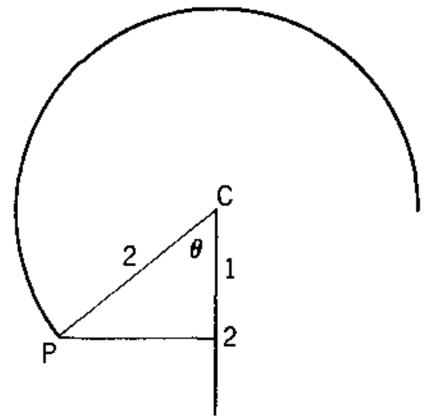
8. The direction of the velocity vector is $\dot{y}/\dot{x}$ or $\sin \omega t/(1-\cos \omega t)$ or $\sin \theta/(1-\cos \theta)$. This direction is also the direction of the tangent as pointed out in (55). Then this exercise is exactly the same as the one in Exercise 8 of Section 5.

9. The magnitude and direction of the acceleration came from $d^2\vec{r}/dt^2$, which is already calculated in Exercise 5. The magnitude is $\sqrt{\ddot{x}^2+\ddot{y}^2}$ and the direction is $\ddot{y}/\ddot{x}$. We find that $|\vec{a}| = 400$. To see that the direction of the acceleration is toward the center we shall prove that it is true for any cycloid in Exercise 10.

10. The direction of the acceleration is $\ddot{y}/\ddot{x}$ and we find that $\ddot{y}/\ddot{x} = \cot \omega t = \cot \theta$. To see that this is the direction of PC of Fig. 18-36, note that P has the coordinates $(R\theta - R\sin\theta,\ R - R\cos\theta)$. Point C has the coordinates $(R\theta, R)$. Then the slope of PC, given by $(y_2-y_1)/(x_2-x_1)$, $= \cot \theta$. Hence the acceleration vector has the direction of PC.

11. At the top of its path P has the y-value of $2R$. Then ωt or $\theta = \pi$. But $|\vec{v}| = 2R\omega$. As pointed out in Exercise 4(b) the center C of the generating circle has a linear velocity to the right of $R\omega$.

12. $d^2s/dt^2 = (\dot{x}\ddot{x} + \dot{y}\ddot{y})/\sqrt{\dot{x}^2 + \dot{y}^2}$. We must show that this is less than $\sqrt{\ddot{x}^2 + \ddot{y}^2}$ or we must show that $\dot{x}\ddot{x} + \dot{y}\ddot{y} < \sqrt{\dot{x}^2 + \dot{y}^2}\ \sqrt{\ddot{x}^2 + \ddot{y}^2}$. If the left side is negative there is no problem. Suppose it is positive and square both sides. Then we must prove that $2\dot{x}\dot{y}\ddot{x}\ddot{y} < \dot{x}^2\ddot{y}^2 + \ddot{x}^2\dot{y}^2$ or $\dot{x}^2\ddot{y}^2 - 2\dot{x}\dot{y}\ddot{x}\ddot{y} + \ddot{x}^2\dot{y}^2 > 0$. But the left side is $(\dot{x}\ddot{y} - \ddot{x}\dot{y})^2$ and so is > 0. If equality holds $\dot{x}\ddot{y} - \ddot{x}\dot{y} = 0$. But then $d(\dot{y}/\dot{x})/dt = 0$ (see (24)) or $d^2y/dx^2 = 0$ and $y = ax + b$.



13. The point moves on a cycloid for which $R = 2$ and $\omega = 20 \cdot 2\pi$ rad. per min. Then we can write the equations of the path by using (5) or we can use Exercise 4(a) to get $|\vec{v}|$ and $|\vec{a}|$. To calculate $|\vec{v}|$ we need the value of ωt or θ when the point is 1 ft. below the center of the wheel. As we see from the figure $\cos\theta = \frac{1}{2}$ and $\theta = \pi/3$. Hence $\omega t = \pi/3$. Then $|\vec{v}| = 2 \cdot 2 \cdot 20 \cdot 2\pi \sin(\pi/6)$ $= 80\,\pi$ ft/min. $|\vec{a}| = 2(40\pi)^2$ $= 3200\,\pi$ ft/min^2.

14. (a) If $\vec{F} = K\vec{d}$, then $F_x = -Kx$ and $F_y = -Ky$. If we let $k = K/m$ and use Newton's second law, which is a vectorial law, then $a_x = d^2x/dt^2 = -kx$ and $a_y = d^2y/dt^2 = -ky$. Hence by solving the two differential equations we find that $x = a_1\cos\sqrt{k}\,t + a_2\sin\sqrt{k}\,t$ and $y = b_1\cos\sqrt{k}\,t + b_2\sin\sqrt{k}\,t$. At $t = 0$, $y = 0$. Hence $b_1 = 0$. Also at $t = 0$, $dx/dt = 0$ so that $a_2 = 0$. Then $x = a_1\cos\sqrt{k}\,t$, $y = b_2\sin\sqrt{k}\,t$. These equations are the parametric equations of the ellipse $(x^2/a_1^2) + (y^2/b_1^2) = 1$.
   (b) The period of the motion depends only on $\sqrt{k}$. It is $2\pi/\sqrt{k}$.

CHAPTER 18, SECTION 8, FIRST SET

1. The parametric equations of the motion are $x = R\cos\omega t$, $y = R\sin\omega t$, where $\omega t = \theta$. In the case of the circle, the linear velocity v is $\dot{s} = R\dot{\theta} = R\omega$ and by (70), $a_T = 0$. Now by (73), $a_N = v^2/R$. But $v^2 = R^2\omega^2$ and the radius of curvature R of the circle is R. Hence $a_N = R\omega^2$.
2. For the circle we may again start with $v = \dot{s} = R\dot{\theta} = R\omega$. Since by (73), $a_T = \ddot{s}$ and ω is a function of t, $a_T = R\,d\omega/dt$. By (73), $a_N = v^2/R$ $= R^2\omega^2/R = R\omega^2$.
3. (a) To obtain $a_T = \ddot{s}$ we need $\dot{s}$. By (56), $\dot{s} = \sqrt{\dot{x}^2 + \dot{y}^2} =$ $\sqrt{R^2\omega^2(1 - \cos\omega t)^2 + R^2\omega^2\sin^2\omega t} = \sqrt{2R^2\omega^2(1 - \cos\omega t)} = 2R\omega\sin(\omega t/2)$. Then $a_T = \ddot{s} = R\omega^2\cos(\omega t/2)$.
   (b) $a_N = v^2/R = v^2K$. Now $v = \dot{s}$ and K is given by (41). We find that $K = (\dot{x}\ddot{y} - \dot{y}\ddot{x})/\dot{s}^3$. Since $v^2 = \dot{s}^2$, $a_N = (\dot{x}\ddot{y} - \dot{y}\ddot{x})/\dot{s} = -R\omega^2\sin(\omega t/2)$.
   (c) Since a_T and a_N are perpendicular, $|a| = \sqrt{a_T^2 + a_N^2} = R\omega^2$.
4. If $v = \dot{s}$ is constant then $a_T = \ddot{s} = 0$. Now a_N is in the perpendicular direction to the tangent and the velocity is in the direction of the tangent. Since a_N is

here the total acceleration the acceleration is perpendicular to the velocity.

5. (a) The parametric equations are $x = Vt$, $y = 16t^2$.
   (b) To calculate $a_T = \ddot{s}$, we need $\dot{s}$. By (56) $\dot{s} = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{V^2 + (32t)^2}$. Then $\ddot{s} = 32^2t/\sqrt{V^2 + (32t)^2}$. $a_N = v^2/R = Kv^2 = (\dot{x}\ddot{y} - \ddot{y}\dot{x})/\dot{s} = 32V/\sqrt{V^2 + (32t)^2}$.
   (c) It should be 32 because the only acceleration acting is that due to gravity. If we calculate $|a| = \sqrt{a_T^2 + a_N^2}$ we get 32.
   (d) At $t = 0$ the x-axis is tangential to the path and the y-axis is normal. Then $a_T = 0$ because there is no horizontal acceleration acting and $a_N = 32$ because gravity acts downward. If we substitute $t = 0$ in (b) we get $a_T = 0$ and $a_N = 32$.

6. From (73) we find that $a_N = 4\pi^2 r/T^2$. On the equator $R = 4000 \cdot 5280$ ft and $T = 1$ day $= 86{,}400$ seconds. Hence $a_N = 0.11$ ft/sec². The tangential acceleration is 0 because an object on the earth's surface moving with the earth rotates at the constant velocity of the earth. Specifically $s = R\theta$, where s is arc traversed along the equator and θ is an angle at the center of the earth which increases as the earth rotates. Then $v = \dot{s} = R\dot{\theta}$. But $\dot{\theta}$ is constant for the earth's rotation. Hence $a_T = \ddot{s} = 0$.

7. (a) According to the text, the satellite must orbit at a velocity v such that the centripetal acceleration required to keep it from falling (or departing from its path), that is, the v^2/r, must equal GM/r^2. Now $GM = 32(4000 \cdot 5280)^2$ and $r = 5000 \cdot 5280$. Then $v^2/r = 20$ ft/sec² approx.
   (b) The earth's gravity exerts just the acceleration required to keep the satellite on its path.

8. (a) The equations are $x = Vt\cos A$, $y = -16t^2 + Vt\sin A$. The tangential velocity $v = \dot{s} = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{V^2 - 64t\sin A + 1024t^2}$.
   (b) All of the velocity is in the direction of the tangent. Hence $v_N = 0$.
   (c) $a_T = \ddot{s}$. Use the result in (a) to calculate $\ddot{s}$. Answer in text.
   (d) $a_N = v^2/R = Kv^2 = (\dot{x}\ddot{y} - \ddot{x}\dot{y})/\dot{s} = (-32V\cos A)/\sqrt{V^2 - 64t\sin A + 1024t^2}$.
   (e) At the maximum height, $\dot{y} = 0$. Then $\dot{s} = \dot{x}$ and $a_N = \ddot{y} = -32$.
   (f) Yes. At the maximum height the normal direction is the vertical direction and we know that the only vertical acceleration acting is due to gravity and that is -32.

9. The example of the text is $x = \cos t + t\sin t$, $y = \sin t - t\sin t$. Now $a_N = Kv^2$. v is the magnitude of the velocity vector and is found in the text to be t. To calculate K we use (40). This gives $K = 1/t$. Then $a_N = t$ which agrees with (76).

CHAPTER 18, SECTION 8, SECOND SET

1. (a) Yes because the velocity will increase and decrease in a manner dependent only on the heights involved.
   (b) No. The velocity depends only on the vertical distance from A to D and D to B.
   (c) We have no proof that the period does not depend on the shape of the curve ADB.
2. The result (78) generalizes the result of Chap. 3, Sec. 4, exercise 5(c). There the motion was along an inclined straight line. For (78) the motion is along any curve but in both cases the velocity depends only on the vertical distance involved.

9. Since we have v_ρ and v_θ the angle α which the velocity vector makes with the radial direction is given by $\tan\alpha = v_\theta/v_\rho$. But the velocity vector is along the tangent and α equals the angle ψ between the radius vector and the tangent. Hence by (22) of Chapter 17, $(1/\rho)d\rho/d\theta = v_\rho/v_\theta$ $= (-1+\sin\theta)/\cos\theta$. Then $\log\rho = \int[(-1+\sin\theta)/\cos\theta]\,d\theta + C$. This integral can be evaluated in various ways. Multiply numerator and denominator by $1+\sin\theta$. Then $\log\rho = \int[(-\cos\theta)/1+\sin\theta)]\,d\theta + C = -\log(1+\sin\theta) + C$. From Fig. 19-7 we see that when $\theta = 0$, $\rho = a$. Then $C = \log a$. Finally $\rho = a/(1+\sin\theta)$. The path is a parabola (see (9) of Chapter 17).
10. We want $\dot\theta$ and $\ddot\theta$. We have that $v_\theta = v\cos\theta$. Then $\dot\theta = v_\theta/\rho = v\cos\theta/\rho$. The equation of the line is $\rho\cos\theta = a$ or $\rho = a/\cos\theta$. Hence $\dot\theta = v\cos^2\theta/a$. Then $\ddot\theta = 2v\cos\theta(-\sin\theta)\dot\theta/a = -2v^2\sin\theta\cos^3\theta/a^2$.
11. We have that $v_\theta = gt\cos\theta$. Then $\dot\theta = v_\theta/\rho = gt\cos\theta/\rho$. From the figure we have $\rho = a/\cos\theta$ so that $\dot\theta = gt\cos^2\theta/a$. To get $\dot\theta$ in terms of t we note that $\tan\theta = AP/OA = gt^2/2a$. Then $\cos\theta = 2a/\sqrt{4a^2+g^2t^4}$. Substituting this in the expression for $\dot\theta$ gives the text's expression for $\dot\theta$. Differentiating $\dot\theta$ with respect to t gives the text's expression for $\ddot\theta$.
12. The text gives the parametric equations. Hence apply (13) and (14). The answers are $a_\rho = -v^2(\cos\theta)/a$ and $a_\theta = -v^2(\sin\theta)/a$.
13. We are given that $\dot\theta = \omega$ = constant. Now a_ρ and a_ρ call for $\dot\rho$ and $\ddot\rho$. But $d\rho/dt = (d\rho/d\theta)(d\theta/dt) = \omega d\rho/d\theta$. Then $\ddot\rho = d\dot\rho/dt = (d\dot\rho/d\theta)(d\theta/dt)$ $= \omega^2 d^2\rho/d\theta^2$. Now substitute in (13) and (14) to get the text's answers.
14. (a) The radial force produces a radial acceleration of $a_\rho = -k\rho$ where ρ is OP. Since it is given that $\dot\theta = \omega$, where ω is constant, we have from (13) that $-k\rho = \ddot\rho - \rho\omega^2$ or $\ddot\rho = -\rho(k-\omega^2)$. This is the equation for simple harmonic motion. See (78) of Chapter 10. Then the period is $2\pi/\sqrt{k-\omega^2}$.

    (b) This part of the problem calls for the equation in ρ and θ. From part (a) we have that $\rho = A\sin\sqrt{k-\omega^2}\,t + B\cos\sqrt{k-\omega^2}\,t$. Let us suppose that $\rho = 0$ when $t = 0$ (though we could have $\rho = \rho_0$ and still derive the result). Then $B = 0$. Since, by part (a), $\dot\theta = \omega$, $\theta = \omega t + C$. Again we can suppose that $\theta = 0$ when $t = 0$ since this says only that the particle is on the polar axis at $t = 0$. Then $\theta = \omega t$. If we substitute this value of t in the expression for ρ and let $\omega^2 = k/2$ we obtain $\rho = A\sin\theta$. This is the polar equation of a circle.
15. We have that $\dot\theta = \omega$. Then $\theta = \omega t$ (whether we add a constant of integration does not matter. Hence $\rho = a(1-\cos\omega t)$. We can now apply (10), (13) and (14) and obtain the text's answers directly.

Solutions to Chapter 19

CHAPTER 19, SECTION 1

1. Solve $\theta = \omega t + \alpha$ for t and substitute in the expression for ρ.
2. Since $t = \theta/\omega$, $\rho = v\theta/\dot{\omega}$.
3. We are given that $\dot{\theta} = \omega$ = constant. Thus $\theta = \omega t$ + constant. Writing the integration constant in the convenient form ωt_0 and using the given equation for ρ, we find $\theta = \omega(t - t_0)$, $\rho = a[1 - \cos\omega(t - t_0)]$.
4. Since $\dot{\theta} = 2$ and $\theta = 0$ when $t = 0$, we have $\theta = 2t$ and then $\rho = e^{2t}$.
5. Since $\dot{\rho} = 3$, we have $\rho = 3t + \rho_0$ and $\theta^2 = 3t + \rho_0$ where ρ_0 is an integration constant.

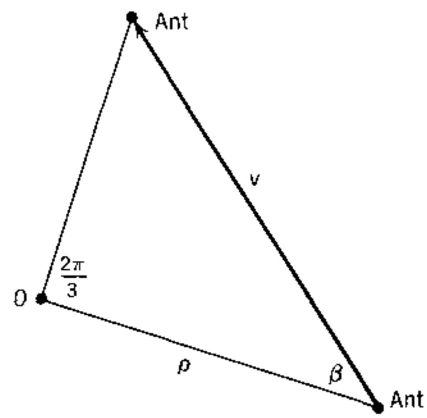
CHAPTER 19, SECTION 2

1. We are told that $\dot{\rho} = v$, $\theta = \pi/6$. Thus by (10), (13), and (14) we find $v_\rho = v$, $v_\theta = 0$, $a_\rho = \dot{v}$, $a_\theta = 0$.
2. (a) Here the equations of motion are $\rho = R$, $\dot{\theta} = \omega$, where R and ω are constant. As in Exercise 2, we find $v_\rho = 0$, $v_\theta = R\omega$, $a_\rho = -R\omega^2$, $a_\theta = 0$.
   (b) They agree because for motion on a circle the radial direction and normal direction coincide (except for sign) and the transverse direction is the tangential direction.
3. We know that $\rho = R$ and $\dot{\theta} = \omega$, but ω is now a function of t. If we use (10), (13), and (14) we obtain the text's answers.
4. From (8) we have that $|d\vec{\rho}/dt| = \sqrt{\rho^2\dot{\theta}^2+\dot{\rho}^2} = \sqrt{\rho^2\dot{\theta}^2+(d\rho/d\theta)\dot{\theta}^2} = \sqrt{\rho^2+(d\rho/d\theta)^2}\ \dot{\theta}$. Now use (33) of Chapter 17. The radical is $ds/d\theta$. Hence the entire expression is ds/dt.
5. Ordinarily ϕ is constant for a particular point P on the line determined by p and α. (These are 5 and $\pi/6$ in Fig. 19-1.) However as ϕ changes with time P moves. We have the parametric equations $\rho = p/\cos ct$ and $\theta = \alpha + ct$. Hence apply (10), (13), and (14). Then $v_\rho = pc\sin\phi/\cos^2\phi$, $v_\theta = pc/\cos\phi$, $a_\rho = pc^2(1 + \cos\phi + \sin^2\phi)/\cos^2\phi$, and $a_\theta = 2pc^2\sin\phi/\cos^2\phi$, where $\phi = ct$.
6. We are given that $\dot{\rho} = v$ and $\dot{\theta} = \omega$, where v and ω are constant. Then $a_\rho = -\rho\omega^2$ and $a_\theta = 2v\omega$.
7. This calls for straightforward application of (10), (13) and (14). Then $v_\rho = -k\rho$, $v_\theta = \omega\rho$, $a_\rho = (k^2 - \omega^2)\rho$, $a_\theta = -2\omega k\rho$, $|v| = \rho\sqrt{\omega^2 + k^2}$, $|a| = \rho(\omega^2 + k^2)$.
8. We must resolve the river's velocity v into its radial and transverse components. If the boat is at P, draw a perpendicular to AP (Fig. 19-7). Then the components are $v\sin\theta$ and $v\cos\theta$. The total radial velocity v_ρ of the boat is then $-v + v\sin\theta$ and the transverse velocity v_θ is $v\cos\theta$.

16. One can take either $\rho = ep/(1 - \cos\theta)$ or $\rho = ep/(1 + e\cos\theta)$ as the equation of the ellipse. [See (6) and (8) of Chapter.17.] The results differ in some signs for one choice as opposed to the other. Since we are given that $\dot{\theta} = \omega$ we have $\theta = \omega t$ (we can take the constant of integration to be 0 since that just means the particle is on the polar axis at $t = 0$). Then, taking say the first equation, $\rho = ep/(1 - e\cos\omega t)$, we now have but to take the text formula for v_ρ, v_θ, a_ρ and a_θ and calculate what they call for. The results are as in the text.

17. The polar coordinate formula for area [(29) of Chapter 17] is $A = \int(\rho^2/2)\,d\theta$ or $dA/d\theta = \rho^2/2$, so that $dA/dt = (dA/d\theta)(d\theta/dt) = \frac{1}{2}\rho^2\dot{\theta}$. Sweeping out equal areas in equal times means $dA/dt = \text{const.}$ Then we see from (14) that $a_\theta = 0$ and this means that the acceleration is totally radial.

18. Since the motions of the three are the same at any instant the three are equally distant from the pole and the angle formed at the pole by the radii vectors to any two is $2\pi/3$. Since each ant moves in the direction of the nearest one with a uniform velocity of 1, the radial component of that velocity is $-1 \cdot \cos\beta$, where β is shown in the Figure. Now $2\beta + (2\pi/3) = \pi$ so that $\beta = \pi/6$. Moreover the radial velocity is $\dot{\rho}$. Hence $\dot{\rho} = -\cos(\pi/6) = -\sin(\pi/3)$. Then $\rho = [-\sin(\pi/3)]t + C$. At $t = 0$, $\rho = 1$. Then $\rho = [-\sin(\pi/3)]t + 1$. When $\rho = 0$ we



find the time to reach the pole, which is where they must meet. The time is $2/\sqrt{3}$.

CHAPTER 19, SECTION 3, FIRST SET

1. If $h = 0$ we see from (19) that $\dot{\theta} = 0$. Then $\theta = \text{const.}$ and the motion is radial.
2. If the force is $f(\rho)\vec{u}_\rho$ then a_θ is necessarily 0 and so $(1/\rho)d(\rho^2\theta)/dt = 0$. Then Kepler's second law follows. See (18) and the steps following it.
3. If $A = (ht/2)+D$ then $dA/dt = k/2$. But $dA/dt = (1/2)\rho^2\dot{\theta}$ so that $h = \rho^2\dot{\theta}$. Then $dh/dt = d(\rho^2\theta)/dt = 0$ because h is a constant. By (14) $a_\theta = 0$ and so the force is radial, that is, directed to a fixed center.
4. From the given equation for ρ we have $1/\rho = (1/ep)+(\cos\theta/p)$. Now $\dot{\rho} = d\rho/dt = (d\rho/d\theta)\dot{\theta} = (d\rho/d\theta)(h/\rho^2) = -hd(1/\rho)/d\theta = h\sin\theta/p$. Then $\ddot{\rho} = (d\dot{\rho}/d\theta)\dot{\theta} = (h\cos\theta/p)(h/\rho^2)$. But $\cos\theta/p = (1/\rho)-(1/ep)$. Hence

$\ddot{\rho}=[(1/\rho)-(1/e\rho)]h^2/\rho^2$. Now $\rho\dot{\theta}^2 = \rho(h^2/\rho^4) = h^2/\rho^3$. Then $\ddot{\rho}-\rho\dot{\theta}^2 = -(h^2/\rho e)(1/\rho^2)$. But $\ddot{\rho}-\rho\dot{\theta}^2 = a_\rho$. Hence the radial acceleration is proportional to $1/\rho^2$.

5. Replace M by the mass of the earth.

CHAPTER 19, SECTION 3, SECOND SET

1. In this case equation (39) [or (41)] applies with $\rho_0 = 12$, $v_0 = 6$ and, in view of (15), $GMm = 120$. Since $m = 2$, $GM = 60$. Then (39) gives the text result at once.
2. As in Exercise 1, $\rho_0 = 12$, and $GM = 60$. Then $e = (\rho_0 v_0^2/GM) - 1$ or $1 - \rho_0 v_0^2/GM$ depending on the value of e which must be positive. With the given values $e = v_0^2/5 - 1$ or $1 - v_0^2/5$.
   (a) For $e = 1/2$, $v_0 = \sqrt{45/6}$ or $\sqrt{15/2}$.
   (b) For $e = 0$, $v_0 = \sqrt{5}$.
   (c) For $e = 1$, $v_0 = \sqrt{10}$.
   (d) For $e = 2$, $v_0 = \sqrt{15}$.
3. For a circle $e = 0$. Then, by (39), $\rho_0 v_0^2/GM = 1$. Also, for circular motion, $v_\rho = 0$. Then $\dot{\rho} = 0$. (If v_0 were 0 the earth would be pulled directly into the sun.)
4. We do have the result that the initial conditions are $\rho_0 v_0^2/GM = 1$. Then for the mass M' half as large as M we would have had $\rho_0 v_0^2/GM' = 2$. By (39) this means that the path would have been parabolic.
5. From (24) by multiplying by $2\dot{\rho}$ we have $2\dot{\rho}\ddot{\rho} = h^2 2\dot{\rho}/\rho^3 - 2GM\dot{\rho}/\rho^2$. We can integrate with respect to t because the left side is the derivative of $\dot{\rho}^2$ and in each of the other terms we have a function of ρ multiplied by $d\rho/dt$. Hence $\dot{\rho}^2 = -h^2/\rho^2 + 2GM/\rho + C$. Since the particle is projected perpendicular to the polar axis, $\dot{\rho}_0$, the value of $\dot{\rho}$ at $\theta = 0$, is 0. Also, by (38) $h = \rho_0 v_0$. Then when $\rho = \rho_0$, our equation gives $0 = -\rho_0^2 v_0^2/\rho_0^2 + 2GM/\rho_0 + C$ or $C = v_0^2 - 2GM/\rho_0$. With this value of C and with $h = \rho_0 v_0$ we get the text result.
6. As is evident from Fig. 19-15 the major axis is the sum of the ρ-values for $\theta = 0$ and $\theta = \pi$. This sum is $\rho_0 + \rho_0^2 v_0^2/(2GM - \rho_0 v_0^2)$. Then $a = 1/2$ of this sum. If we add and take $1/2$ we get the text result. The same result holds for (41).

7. From (10) we have that $v^2 = \dot{\rho}^2 + \rho^2\dot{\theta}^2$. From (33) we find that $\dot{\rho} = e \sin(\theta + \alpha)\dot{\theta}\rho/[1 + e \cos(\theta + \alpha)]$. Then $v^2 = [\{e^2 \sin^2(\theta + \alpha)/[1 + e \cos(\theta + \alpha)]^2\} + 1]\ \dot{\theta}^2\rho^2$. Now by (19) $\dot{\theta} = h/\rho^2$. Hence $v^2 = \{[1 + 2e \cos(\theta + \alpha) + e^2]/[1 + e \cos(\theta + \alpha)]^2\}\ (h^2/\rho^2)$. By (33) $v^2 = (G^2M^2/h^2)[1 + 2e \cos(\theta + \alpha) + e^2]$. This is the first form of v^2 in the text. Now if we write the bracket as $2 + 2e \cos(\theta + \alpha) + e^2 - 1$ and write $v^2 = (G^2M^2/h^2)[2 + 2e \cos(\theta + \alpha)] + (G^2M^2/h^2)(e^2 - 1)$ and use (33) we obtain the second expression for v^2.

8. To determine the points on the conic where v is a maximum or minimum it is easier to use the first result in Exercise 7, $v^2 = (G^2M^2/h^2) \cdot [1 + 2e \cos(\theta + \alpha) + e^2]$. We wish to locate the θ-values at which maxima or minima occur. Hence $2v(dv/d\theta) = (G^2M^2/h^2)(-2e \sin(\theta + \alpha))$. Now $dv/d\theta = 0$ when $2e \sin(\theta + \alpha) = 0$, so that $\theta + \alpha = 0$ or $\theta + \alpha = \pi$. The result in Exercise 7 presupposes the equation (33) and this means geometrically the situation portrayed in Fig. 19-13. Hence the possible minima and maxima occur on the radii vectors $\theta = -\alpha$ and $\theta = \pi - \alpha$. One could consider this answer sufficient.

To decide whether either of these θ-values does indeed furnish a maximum or a minimum calls for a separate discussion of ellipse, parabola and hyperbola. Thus in the case of the ellipse, the two values of θ are the locations of the vertices at the ends of the major axis. We know from Kepler's second law that the radius vector to the points of the ellipse must sweep out equal areas in equal times. This physical fact tells us that $\theta = -\alpha$ must be a point of maximum velocity and $\theta = \pi - \alpha$ must be a point of minimum velocity. To check this mathematically we can test whether $dv/d\theta$ changes from positive to negative or the reverse. We can take v to be positive (because this is ds/dt). Take α to be, for example, 30°. Then as θ passes through the value $-30°$, say from $-35°$ to $-25°$, $dv/d\theta$ changes from + to − . Hence $\theta = -\alpha$ is a maximum.

The discussion for the parabola and hyperbola are similar except that $\theta = \pi - \alpha$ is not a point on the parabola because, see (33) or (35), e is 1 for the parabola and so θ is infinite. For the hyperbola only one branch, that enclosing the pole, is actually involved in the motion and in this case, too, $\theta = -\alpha$ furnishes a maximum. The value $\theta = \pi - \alpha$ locates the vertex of the other branch and this value furnishes a minimum.

9. (a) The Exercise calls, first of all, for rederiving (33) to obtain the equation of a conic. The only change, however, is that in (24) we must write $+GM/\rho^2$ in place of $-GM/\rho^2$. The reason is that the force is repulsive and is in the direction of increasing ρ. Consequently we obtain in place of (28), $d^2u/d\theta^2 + u = -GM/h^2$ and, in place of (30), $u = C \cos(\theta + \alpha) - GM/h^2$. Then (33) becomes $\rho = -(h^2/GM)/[1 - e \cos(\theta + \alpha)]$. We must now determine e. We repeat steps (38) to (39). Because the signs in our equation differ from those in (33) we obtain $e > 1$. The conic is a hyperbola and the particle will move on one branch.

To show that the branch is the one which does not contain the center of force one must make a distinction which we have ignored in the text. The equation of the hyperbola, whether it be of the form (33) of the text or of the above form represents both branches. One branch is given by values of θ for which ρ is positive and the other by values of θ for which ρ is negative. In the present example we want the branch for which ρ is positive because we took the acceleration to be positive or in the direction of increasing ρ. This is a repulsive acceleration when ρ is positive but an attractive acceleration when ρ is negative (because it reduces the numerical value of ρ). We may, for the purpose of seeing what branch of our hyperbola has positive ρ-values take α to be 0 in our equation. It will be found by graphing an equation of the form $\rho = -ep/(1 - e\cos\theta)$, with $e > 1$ that the branch with positive ρ-values is the one which does not contain the pole or center of force.

(b) We see from the expression for e above that when $v = \sqrt{GM/\rho_0}$ then $e = 2$.

10. To take advantage of the earlier result we write $ds/dt = (ds/d\theta)(d\theta/dt) = (\rho^2/d)\dot{\theta}$. By (19) $\dot{\theta} = h/\rho^2$. Hence the result.

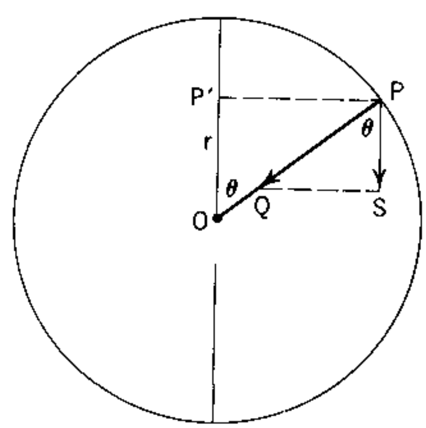
11. Equations (17) and (19) still hold. In fact, except for the fact that $-GM/\rho^2$ is replaced by $-k\rho$ we may use the theory right through except that (28) now becomes $(d^2u/d\theta^2) + u = k/h^2u^3$. We use the old device of multiplying through by $2\,du/d\theta$. Then every term is, integrable and we obtain $(du/d\theta)^2 + u^2 = -k/h^2u^2 + 2C$. (we use 2C for later convenience.) Then $(du/d\theta)^2 = (-g/u^2) - u^2 + 2C = (-g - u^4 + 2Cu^2)/u^2$ where $g = k/h^2$. Then $d\theta/du = u/\sqrt{-g - (u^4 - 2Cu^2)} = u/\sqrt{-g + C^2 - (u^2 - C^2)^2}$. Let $w = u^2 - C^2$. Then $dw/du = 2u$. Hence our integrand can be handled by formula 13 in the Table of Integrals. We get $\theta + D = \frac{1}{2}\sin^{-1}(w/\sqrt{C^2 - g})$ or $w = \sqrt{C^2 - g}\,\sin(2\theta + 2D)$. Now $u^2 = C^2 + \sqrt{C^2 - g}\,\sin(2\theta + 2D)$ and since $u = 1/\rho$ we get the text result. We can change to cosine by changing 2D to another constant. As far as student solution is concerned, it would be satisfactory to have the student show that $u = 1/\rho = \sqrt{A + B\cos(2\theta + \alpha)}$ satisfies $d^2u/d\theta^2 + u = k/h^2u^3$ for proper choice of A and B. It turns out that $A^2 - B$ must equal k/h^2.

12. If we compare the second expression in Exercise 7 with the desired result then we see that we need show only that $(GM/h^2)(e^2 - 1) = -1/a$. We have from Exercise 6 that $1/a = (2GM - v_0^2\rho_0)/GM\rho_0 = 2/\rho_0 - v_0^2/GM$. We have $h = \rho_0 v_0$ and from (39) we have $e^2 - 1 = \rho_0 v_0^2(\rho_0 v_0^2 - 2GM)/G^2M^2$. If we substitute the values of h and $e^2 - 1$ in $(GM/h^2)(e^2 - 1)$ we find that in the light of Exercise 6, the quantity $(GM/h^2)(e^2 - 1) = -1/a$.

13. At the point closest to the sun $d\rho/d\theta = 0$ and since $d\rho/d\theta = (d\rho/dt)/(d\theta/dt)$, then $\dot{\rho} = 0$. Then the velocity is entirely transverse. This is the case for which (39) holds and (39) is the desired answer.

CHAPTER 19, SECTION 4

1. If the orbit is circular then it is given by $\rho = \rho_0$. Then $\dot{\rho} = 0$. Also by (19) $h = \rho^2\dot{\theta} = \text{const.}$ Since ρ is a constant, so is $\dot{\theta}$.
2. This follows at once from (46) where a is now ρ_0.
3. We again use (46), that is, $T^2 = 4\pi^2 a^3/GM$ where T is now 24 hours and a is the desired distance. For GM we can use $32R^2$ where R is the radius of the earth. If distances are in feet, T must be in seconds. Straightforward calculation gives the answer in the text.
4. Use (46), namely $T^2 = 4\pi^2a^3/GM$. We can measure T. We know a and G is a universal constant which is known. Hence we can find M.
5. $T = 2\pi\rho_0^{3/2}/\sqrt{GM}$. $\rho_0 = 4300 \cdot 5280$ and $GM = 32 \cdot (4000 \cdot 5280)^2$. Calculate T.
6. In this case $v_0 < \sqrt{GM/\rho_0}$. This is discussed in (41). The point corresponding to $\theta = 0$ is farthest from the center of force which in the present case is the center of the earth.
7. The satellite is closest to the center when $\theta = \pi$. Hence use (41) with $\theta = \pi$. This gives the result immediately.
8. We must fix v_0 so that the point on the path nearest the earth is greater than R. That is, from Exercise 7, $\rho_0^2 v_0^2/(2GM - \rho_0 v_0^2) > R$. Now the trajectory is an ellipse so that, by (40a), $2GM > \rho_0^2 v_0^2 > \rho_0 v_0^2$. Then we may multiply both sides of the original inequality by $2GM - \rho_0 v_0^2$ and by simple steps establish the desired inequality.
9. $\rho_0 = 5000 \cdot 5280$; $v_0 = 2000$. Then $\rho_0 v^2 < GM$. Then (41) applies and we may substitute the values in it. The arithmetic gives the result in the text.
10. With the given condition on v_0 the path is elliptical and (39) applies. The point corresponding to $\theta = \pi$ gives the point farthest from the earth. Let $\theta = \pi$ in (39) and we obtain the text's result.
11. The centripetal acceleration of the projectile is $PQ = GM/R^2$. The vertical component is $PS = PQ\cos\theta$. However $\cos\theta = r/PO = r/R$ where r is the corresponding position of the object in the tunnel. Hence the vertical component of the centripetal acceleration is $(GM/R^2)(r/R) = GMr/R^3$. If we check (73) of Chapter 10 we see that the acceleration acting on the object in the tunnel is GMr/R^3. This means that though the projectile moves in a circular path with a radial or centripetal acceleration of GM/R^2, its motion is equivalent to that of an object which moves only vertically with an acceleration of GMr/R^3. Since the object in the tunnel has the latter acceleration its motion takes the same time as the projectile motion. Note that if the projectile is fired from the North Pole, say, its vertical velocity at that point is 0, which is the case for the object placed in the tunnel and just released. Hence the initial velocities are the same.



Solutions to Chapter 20

CHAPTER 20, SECTION 2

1. Let $\sin x = c_0 + c_1 x$. Then for $x = 0$, we obtain $c_0 = 0$. Differentiate. Then $\cos x = c_1$. At $x = 0$, $c_1 = x$. Hence $\sin x = x$ approximately for x near 0.
2. We use (7) as a guide with $a = \pi/4$ and $n = 4$. Then the approximating polynomial $g(x)$ is given by letting $f(x)$ be $\sin x$. The answer is in the text.
3. Set $e^x = c_0 + c_1 x + \ldots + c_n x^n$. Let $x = 0$. Then $c_0 = 1$. Differentiate both sides. This gives $e^x = c_1 + 2c_2 x + \ldots + nc_n x^{n-1}$. Let $x = 0$. Then $c_1 = 1$. Differentiate again. Then $e^x = 2c_2 + 2 \cdot 3c_3 x + \ldots + n(n-1)x^{n-2}$. Let $x = 0$. Then $c_2 = \frac{1}{2}$. Repeat this. We see that $c_i = 1/i!$ for $i = 0, 1, 2, \ldots, n$. Note that by definition $0! = 1$.
4. Start with $\sin x = c_0 + c_1 x + \ldots + c_n x^n$ and carry out the process of Exercise 3.

CHAPTER 20, SECTION 3

1. (a) Use (20) with $f(x) = \cos x$, $a = \pi/4$ and $n = 2$. Answer in text.
   (b) Use (22) as a guide with $f(x) = \cos x$. However every odd derivative will be 0. Hence if we want three non-zero terms we must let $n = 4$ and we get $\cos x = 1 - (x^2/2!) + (x^4/4!) - \sin\mu(x^5/5!)$. Note that $\sin x = 0$ at $x = 0$ and so if we carry the expansion to one more term, the $x^5/5!$ term drops out and we have as the last term $-\cos\mu(x^6/6!)$.
   (c) Use (20) with $f(x) = \cos x$, $a = \pi/4$ and $n = 2$. The answer is in the text.
   (d) Use (22) with $f(x) = \sin x$. However every even derivative will be 0. Hence let $n = 5$. Then $\sin x = x - (x^3/3!) + (x^5/5!) - \sin\mu(x^6/6!)$. As in (b) we could use the fact that $\sin x = 0$ at $x = 0$ and go one more term in the expansion. Then the $x^6/6!$ term drops out and the last term becomes $-\cos\mu(x^7/7!)$.
   (e) Use (20) with $f(x) = \tan x$, $a = \pi/4$, and $n = 2$. Answer in the text.
   (f) Use (20) with $f(x) = e^x$, $a = 1$, and $n = 2$. Then $e^x = e + e(x-1) + e(x-1)^2/2! + e^\mu(x-1)^3/3!$
2. For $n = 0$, we have by (20), that $f(x) = f(0) + f'(\mu)x$. Thus $x^4 = 4\mu^3 x$ and at $x = 1$, $\mu = 1/\sqrt[3]{4}$. Similarly for $n = 1$, we obtain $x^4 = 12\mu^2(x^2/2)$ and at $x = 1$, $\mu = 1/\sqrt{6}$. For $n = 2$, $x^4 = 24\mu(x^3/6)$ and at $x = 1$, $\mu = \frac{1}{4}$. For $n = 3$, $x^4 = 24x^4/24$. The values of μ are decreasing. For the case of $n = 3$, since the fourth derivative is constant any value of μ would do.

(c) By comparison with the p-series, the series converges.

(d) Since $1/\sqrt{n^3} = 1/n^{3/2}$, the series converges by comparison with the p-series.

(e) This is the p-series with $p = \frac{1}{2}$; hence it diverges.

(f) Note that $1/n(n+1) < 1/n^2$ and use the p-series; hence the series converges.

(g) Note that $2n/(n+1)(n+2) > 1/(n+2)$ and use the harmonic series $\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$ for comparison. Neglect of the first two terms in the harmonic series does not affect divergence.

(h) If we factor $\frac{1}{3}$ out of all the terms we have $\frac{1}{3}$ of the harmonic series. But the harmonic series diverges or its "sum" becomes infinite. Hence the given series diverges.

(i) The "sum" of the given series is the "sum" of the harmonic series minus the quantity $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$. Since the harmonic series diverges, so does the given series.

(j) $\sqrt[n]{3} > 1$ but approaches 1 as n becomes infinite. Hence the n-th term of the given series does not approach 0 and the series diverges.

(k) $1/(n^2 + 1) < 1/n^2$. The latter is the n-th term of the p-series with $p = 2$ and so converges. Hence by the comparison test the given series converges.

(l) The sum is $\frac{1}{2}$ the "sum" of the harmonic series. Since the latter diverges so does the given series.

(m) The terms of this series are larger term for term than the terms in (l). The latter diverges; hence so does the former.

(n) Since $\log n < n$, the terms of the given series are greater than the terms of the harmonic series. Hence the given series diverges.

(o) The terms are $\frac{3}{2}$ of the corresponding terms of the harmonic series. The latter diverges. Hence so does the former.

2. The first comparison is fallacious since the next terms in the given series are $\frac{1}{14} + \frac{1}{17} + \frac{1}{20} + \ldots$ while the next terms in the geometric series are $\frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \ldots$ and the terms of the first are not less than the terms of the second series. To avoid being fooled by the coincidence that the first few terms of a given series are less than or greater than the first few terms of a comparison series, one should always, if possible, consider the general term. In the present case the first comparison claims that $1/(3n-1) < 1/2^n$ which is obviously false for large n and the second comparison claims $1/(3n-1) > 1/3^n$ which is correct for all n.

3. Again the comparison is fallacious because $1/n(n+1) > 1/2^n$ for n large. In the present case the given series actually does converge (see 1(f)), but the argument given does not establish this fact.

4. The argument is correct and does show that the harmonic series diverges provided that the grouping of terms is correct. We did find cases where grouping is not correct (Section 6, Exercises 7 and 8). However for a series

CHAPTER 20, SECTION 4

1. In Exercise 1(b) of the preceding section we found that $\cos x = 1 - x^2/2! + x^4/4! - \cos\mu(x^6/6!)$. Hence let $x = 0.1$ in the first three terms. The error is given by the remainder, $\cos\mu(x^6/6!)$. Since $|\cos\mu| < 1$, the remainder is less than $(.1)^6/6!$ and this last is $1.4 \cdot 10^{-9}$.
2. The error is given by $\cos\mu(x^7/7!)$. Since $|\cos\mu| < 1$, $x^7/7!$ for $x = 0.2$ is $(.2)^7/7!$
3. Using the procedure of the preceding set of Exercises we get $e^x = 1 + x + x^2/2 + x^3/6 + e^\mu(x^4/24)$. Then from the first four terms we get $e^{1/2} = 1.65 + R$ where $R = e^\mu(1/16 \cdot 24)$. Since μ is between 0 and $1/2$, $e^\mu < e^{1/2}$. Now $e < 3$; hence $e^{1/2} < \sqrt{3}$ and $R < \sqrt{3}/16 \cdot 24 < 0.005$.
4. For $f(x) = \log(1 + x)$, $f'(x) = 1/(1 + x)$, $f''(x) = -1/(1 + x)^2$, etc. Evaluate these derivatives at $x = 0$. Then $\log(1 + x)$ expanded around $x = 0$ is $\log(1 + x) = \log 1 + 1x - x^2/2! + x^3/3! - x^4/4! + [24/(1 + \mu)^5](x^5/5!)$. Now let $x = 1/2$ and calculate the sum of the first four terms ($\log 1 = 0$). This sum is 0.401. Now $R = [24(1 + \mu)^5](1/2^5 \cdot 5!)$. Since $\mu > 0$ we may say that $r < 1/160$.
5. This is just an extension of Exercise 1. Carry two more terms in the expansion.

CHAPTER 20, SECTION 5

1. Apply (29) in each case.
   (a) $\sin x = x - x^3/3! + x^5/5! - x^7/7! + \dots$.
   (b) $\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + \dots$.
   (c) $\log(1 + x) = x - x^2/2 + x^3/3 - x^4/4 + \dots$.
   (d) $e^x = 1 + x + x^2/2! + x^3/3! + \dots$.
   (e) $e^{-x} = 1 - x + x^2/2! - x^3/3! + \dots$.
   (f) $\tan x = x + x^3/3 + 2x^5/15 + 17x^7/315 + 62x^9/2835 + \dots$.
2. (a) Use (28) with $a = \pi/4$ and $f(x) = \sin x$. Then $\sin x = (1/\sqrt{2})[1 + (x - \pi/4) - (x - \pi/4)^2/2! - (x - \pi/4)^3/3! + (x - \pi/4)^4/4!) + \dots]$.
   (b) $e^x = e + e(x - 1) + e(x - 1)^2/2! + e(x - 1)^3/3! + \dots$.
   (c) $\log(1 + x) = \log 3 + (x - 2)/(3 \cdot 1) - (x - 2)^2/(3^2 \cdot 2!) + (x - 2)^3/3^3 \cdot 3!) - \dots$.

CHAPTER 20, SECTION 6

1. (a) Geometric series with $r = 1/3 < 1$; hence convergent.
   (b) Geometric series with $r = 2 > 1$; hence by (33) $S_n = 2^n$ which diverges.
   (c) Geometric series with $r = .1 < 1$; hence convergent.
   (d) Geometric series with $r = 1/4 < 1$; hence convergent.
   (e) Geometric series with $r = .3 < 1$; hence convergent.
   (f) $S_n = n(n + 1)/2$ which diverges.
   (g) $S_n = .01[n(n + 1)/2]$ which diverges.
2. (a) For $r = 1$, the geometric series has the form $a + a + a + a + \ldots$ which clearly diverges.
   (b) For $r = -1$, the geometric series has the form $a - a + a - a + a - \ldots$ which, the reasoning used in examining (39) shows, diverges.
3. If $r > 1$, we can write r as $1 + h$. Then $r^n = (1 + h)^n = 1 + nh +$ positive terms As n becomes infinite so does r^n. Then, since $S_n = (ar^n - a)/(r - 1)$, S_n becomes infinite.
4. It diverges because, supposing the first term to be positive, the odd partial sums become larger and larger and the even partial sums become smaller and smaller. Hence all the partial sums do not approach one number.
5. (a) $|r| = 1/2 < 1$; hence it converges. (b) $|r| = 1/3 < 1$; hence it converges. (c) $|r| = 3/2 > 1$; hence it diverges.
6. The argument assumes that the series has a sum S. If this were the case then 3 would indeed be -1. One can make a sensible argument by saying, "Suppose $S = 1 + 2 + 4 + \ldots$. Then by the steps in the text $S = -1$. Since a sum of positive terms cannot be negative, the assumption that the series has a sum is incorrect."
7. The grouping of the terms in the form $S = (1 - 1) + (1 - 1) + \ldots$ gives only the even-numbered partial sums of the original series. But if the original series is to converge all the partial sums must approach the same number. This example show us that grouping of terms may lead to the wrong conclusion.
8. Here the grouping of the terms gives us only the odd-numbered partial sums. If the original series is to converge all the partial sums must approach the same number.
9. The argument assumes that the series has a sum. If it did the sum would be $1/2$. But we have no argument to show that the original series has a sum. In fact we know from the text that it does not.

CHAPTER 20, SECTION 7

1. (a) Since $(2^{n-1} + 1)/2^n > 1/2$ and the series $1/2 + 1/2 + 1/2 + \ldots$ diverges, the given series diverges.
   (b) Since $1/2^{n+1} < 1/2^n$, the given series converges by comparison with the geometric series with $a = 1$, $r = 1/2 < 1$.

of positive terms the grouping does not alter the convergence or divergence. The grouping does give only the even-numbered partial sums. However the odd-numbered partial sums lie between the even-numbered ones. That is, $S_1 < S_2 < S_3 < S_4 < S_5 \ldots$. If the even-numbered ones approach a fixed number so do the odd-numbered ones and if the even-numbered ones become infinite, so do the odd-numbered ones.

CHAPTER 20, SECTION 8

1. (a) The series of absolute values is the p-series with $p = 3$. Hence the series converges absolutely.
   (b) The series of positive and negative terms satisfies the conditions of the first theorem of the section. Hence it converges. However the series of absolute values is greater term for term than the series whose n-th term is $1/3n$ and the latter is $\frac{1}{3}$ of the harmonic series and so diverges. Hence the original series is conditionally convergent.
   (c) The n-th term does not approach 0. Hence the series is divergent.
   (d) Divergent for the same reason as in (c).
   (e) The series of absolute values is less, term for term, than the p-series with $p = 2$. Hence the series converges absolutely.
   (f) The series of positive and negative terms converges because it meets the conditions of the first theorem in the section. However the series of absolute values is twice the harmonic series and so diverges. The original series is conditionally convergent.
2. (a) No. See the first theorem in Section 7.
   (b) Yes, for example, $1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{8} - \frac{1}{8} + \frac{1}{16} - \frac{1}{16} + \ldots$.

CHAPTER 20, SECTION 9

1. (a) $\lim_{n\to\infty} |[(n + 1)/3^{n+1}]/(n/3^n)| = \lim_{n\to\infty} (n + 1)/3n = \frac{1}{3} < 1$; hence the series is convergent. To see the limit more clearly divide numerator and denominator by n.
   (b) $\lim_{n\to\infty} |[2^{n+1}/(n + 1)]/(2^n/n)| = \lim_{n\to\infty} 2n/(n + 1) = 2 > 1$; the series is divergent.
   (c) $\lim_{n\to\infty} |[2^{n+1}/(n + 1)^2]/(2^n/n^2)| = \lim_{n\to\infty} 2n^2/(n^2 + 2n + 1) = 2 > 1$; the series is divergent.
   (d) $\lim_{n\to\infty} |[(n + 1)/10^n]/(n/10^{n-1})| = \lim_{n\to\infty} (n + 1)/10n = \frac{1}{10} < 1$; the series is convergent.
   (e) $\lim_{n\to\infty} |[(n + 1)(2/3)^{n+1}]/n(2/3)^n| = \lim_{n\to\infty} (2n + 2)/3n = \frac{2}{3} < 1$; the series is convergent.

(f) $\lim\limits_{n\to\infty} |[(n+1)!/10^{n+1}]/(n!/10^n)| = \lim\limits_{n\to\infty} (n+1)/10 = \infty > 1$; the series is divergent.

(g) $\lim\limits_{n\to\infty} |(2^{n+1}/\sqrt{n+1})/(2^n/\sqrt{n})| = \lim\limits_{n\to\infty} 2\sqrt{n}/\sqrt{n+1} = 2 > 1$; the series is divergent.

(h) $\lim\limits_{n\to\infty} |[(n+1)!/3\cdot 5\cdot 7\ldots(2n+3)]/[n!/3\cdot 5\cdot 7\ldots(2n+1)]| = \lim\limits_{n\to\infty}$ $(n+1)/(2n+3) = \frac{1}{2} < 1$; the series is convergent.

(i) $\lim\limits_{n\to\infty} \left|\dfrac{1\cdot 3\cdot 5\ldots(2n+1)/1\cdot 4\cdot 7\ldots(3n+1)}{1\cdot 3\cdot 5\ldots(2n-1)/1\cdot 4\cdot 7\ldots(3n-2)}\right| = \lim\limits_{n\to\infty} \dfrac{2n+1}{3n+1} = \dfrac{2}{3} < 1$; the series is convergent.

(j) $\lim\limits_{n\to\infty} |[1/(2n+1)(2n+2)]/[1/(2n-1)2n]| = \lim\limits_{n\to\infty} n(2n-1)/(n+1)(2n+1)$ $= 1$; hence no test.

CHAPTER 20, SECTION 10

1. (a) $|a_{n+1}/a_n| = [1/(n+1)^2]/(1/n^2) = n^2/(n+1)^2$. Hence the limit is 1 and the series certainly converges for $-1 < x < 1$. For $x = -1$ we have the negative of the p-series for $p = 2$ and this converges. For $x = 1$ we have the conditions for an alternating series to converge. Hence the interval is $-1 \le x \le 1$.

   (b) $|a_{n+1}/a_n| = [1/(2n)!]/[1/(2n-2)!] = 1/(2n-1)(2n)$. Hence the limit is 0 and the series converges for $-\infty < x < \infty$.

   (c) $|a_{n+1}/a_n| = [1/(n+1)]/(1/n) = n/(n+1)$. Hence the limit is 1 and the series certainly converges for $-1 < x < 1$. For $x = 1$ the series is an alternating series and meets the conditions for convergence. For $x = -1$, the series is the negative of the harmonic series and so diverges. Hence the interval is $-1 < x \le 1$.

   (d) Since this is not a power series it is best to apply the ratio test directly. We have $x^{(n+1)^2}/x^{n^2} = x^{2n+1}$. The limit as n becomes ∞ must be < 1 for the series to converge. But this will hold for $|x| < 1$. For $x = 1$ the series certainly diverges and for $x = -1$, the series is $1 - 1 + 1\ldots$ and diverges. Hence the interval is $-1 < x < 1$.

   (e) $|a_{n+1}/a_n| = \sqrt{n+1}/\sqrt{n}$. The limit is 1. Hence so far we have convergence for $-1 < x < 1$. For $x = 1$ we have a p-series with $p = \frac{1}{2}$. Hence divergence. For $x = -1$ we have a convergent alternating series. Hence the interval is $-1 \le x < 1$.

   (f) $|a_{n+1}/a_n| = (n-1)!/n! = 1/n$. The limit is zero. Hence the series converges for all values of x.

   (g) $|a_{n+1}/a_n| = (2n-2)!/2n! = 1/2n(2n-1)$. The limit is zero. Hence the series converges for all values of x.

(h) $|a_{n+1}/a_n| = (n+1)/n$. The limit is 1. Hence so far the interval of convergence is $-1 < x < 1$. For $x = 1$ and for $x = -1$ the n-th term does not approach 0. Hence the interval is $-1 < x < 1$.

(i) $|a_{n+1}/a_n| = n^2/(n-1)^2$. The limit is 1. Hence so far $-1 < x < 1$. For $x = 1$ we have the p-series with $p = 2$ and so convergence. For $x = -1$ we can rely upon the absolute convergence when $x = 1$. Hence the interval is $-1 \le x \le 1$.

(j) $|a_{n+1}/a_n| = [(n+1)/n!]/[n/(n-1)!] = (n+1)/n^2$. The limit is 0. Hence the series converges for all x.

(k) $|a_{n+1}/a_n| = [2^{n+1}/(n+1)^2 + 1]/[2^n/(n^2+1)] = 2(n^2+1)/[(n+1)^2+1]$. The limit is 2. Hence so far the interval is $-\frac{1}{2} < x < \frac{1}{2}$. For $x = \frac{1}{2}$ we have terms which are less than those of the p-series for $p = 2$. Hence convergence. For $x = -\frac{1}{2}$ we can rely upon the absolute convergence when $x = \frac{1}{2}$. Hence $-\frac{1}{2} \le x \le \frac{1}{2}$.

(l) $|a_{n+1}/a_n| = [1/2^n(n+1)]/(1/2^{n-1}n) = n/2(n+1)$. The limit is $\frac{1}{2}$. Hence so far the interval is $-2 < x < 2$. For $x = 2$ we have the harmonic series. For $x = -2$ we have a convergent alternating series. Hence $-2 \le x < 2$.

(m) $|a_{n+1}/a_n| = [(n+1)/2^n3^{n+1}]/(n/2^{n-1}3^n) = (n+1)/6n$. The limit is $\frac{1}{6}$. Hence so far $-6 < x < 6$. For $x = 6$ the series is $\frac{1}{3} + \frac{2}{3} + \frac{3}{3} + \ldots + n/3 + \ldots$. This certainly diverges because the n-th term does not approach 0. For $x = -6$, the same is true. Hence $-6 < x < 6$.

(n) $|a_{n+1}/a_n| = [1/2^n(n+2)]/[1/2^{n-1}(n+1)] = (n+1)/2(n+2)$. The limit is $\frac{1}{2}$. Hence so far the interval is $-2 < x < 2$. For $x = 2$, the n-th term is $2/(n+1)$. This is twice the harmonic series. For $x = -2$, the series is alternating and meets the convergence conditions. Hence $-2 \le x < 2$.

2. (a) Replace $x - 1$ by t temporarily. Then $\lim_{n\to\infty} |a_{n+1}/a_n| = 1$. For $t = 1$ or -1, the n-th term does not approach 0. Hence the interval is $-1 < t < 1$ or $0 < x < 2$.

   (b) Replace $x - 2$ by t temporarily. Then $\lim_{n\to\infty} |a_{n+1}/a_n| = 0$. Hence the series converges for all t. This can be stated as $-\infty < t < \infty$. Hence the series converges for $-\infty < x < \infty$.

   (c) Replace $x + 4$ by t temporarily. Then $\lim_{n\to\infty} |a_{n+1}/a_n| = 0$. Hence, as in (b), the series converges for all values of x.

   (d) Replace $x + 3$ by t. Then $\lim_{n\to\infty} |a_{n+1}/a_n| = 1$. Hence the series converges for $-1 < t < 1$ and since $x = t - 3$, the series converges for $-4 < x < 1$.

3. A power series may converge for say, $-a \le x < a$. To be absolutely convergent it would have to converge for $x = a$ also. If one excludes the end values of the interval of convergence then in the (open) interval it is absolutely convergent. See the theorem on p.

CHAPTER 20, SECTION 11, FIRST SET

1. (a) The Maclaurin series is given by (29) or (80) where n is allowed to become infinite. The series is given in the text. Use of the ratio test shows that it converges for all x.
   (b) The method is as in (a), $\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + \ldots$. Again by the ratio test the series converges for all x.
   (c) By applying (29) or (80) where n is infinite we get the series in the text. By applying the ratio test we get that the series converges for $-1 < x < 1$. For $x = 1$ we get a convergent alternating series. For $x = -1$ we get the negative of the harmonic series which is divergent.
2. (a) $R_n = \sin \mu x^n/n!$ (or $\cos \mu$). Now $|\sin \mu| \leq 1$. Hence we have to be concerned only with $x^n/n!$ Now use the argument on p. 141. That is, given any value of x choose an m such that $m > 2|x|$. Then for any $n > m$, $|x^n/n!| = (x^m/m!)[x/(m+1)]\cdots x/n$. But for $n > m$, $n > 2x$ or $x/n < \frac{1}{2}$. Hence $|x_n/n!| < (x^m/m!)(1/2^{n-m})$. Now m is fixed and so $x^m/m!$ is fixed but as n becomes infinite, $1/2^{n-m}$ approaches 0. Hence $|x^n/n!|$ approaches 0. Then R_n approaches 0 for any given x.
   (b) The argument is exactly the same.
3. A series may represent a function in only a part of the domain for which the function is defined.
4. Using (87) we find that $(1 + \frac{2}{27})^{1/3} = 1 + \frac{2}{81} - (\frac{2}{81})^2 = 1.0244$ approx. Then $\sqrt[3]{29} = 3(1.0244)$ approx.
5. (a) $\sqrt[3]{30} = \sqrt[3]{3 + 27} = 27^{1/3}(1 + \frac{3}{27})^{1/3}$ and apply (87).
   (b) $\sqrt[4]{18} = \sqrt[4]{2 + 16} = 2(1 + \frac{1}{8})^{1/4}$. Apply (87). Ans. 2.060.
   (c) $\sqrt[5]{34} \quad \sqrt[5]{2 + 32} = 2(1 + \frac{1}{16})^{1/5}$. Apply (87).
   (d) $\sqrt[4]{8} = \sqrt[4]{16 - 8} = 2(1 - \frac{1}{2})^4$. Apply (87). Ans. 1.7.
   (e) $\sqrt[4]{75} = \sqrt[4]{81 - 6} = 3(1 - \frac{2}{27})^{1/4}$. Apply (87).
   (f) $\sqrt[3]{25} = \sqrt[3]{27 - 2} = 3(1 - \frac{2}{27})^{1/3}$. Apply (87). Ans. 2.924.

CHAPTER 20, SECTION 11, SECOND SET

1. $\log(1 + x) = \int_0^x dt/(1 + t) = \int_0^x (1 - t + t^2 - t^3 + \ldots)\,dt = x - x^2/2 + x^3/3 - \ldots$.
2. $\sin x^2 = x^2 - x^6/3! + x^{10}/5! - \ldots$. Then $\int_0^x \sin t^2\,dt = \int_0^x (t^2 - t^6/3! + t^{10}/5! - \ldots)\,dt$ $= x^3/3 - x^7/7\cdot 3! + x^{11}/11\cdot 5! - \ldots$.
3. (a) $\arctan x = \int_0^x dt/(1 + t^2) = \int_0^x (1 + t^2)^{-1}\,dt = \int_0^x (1 - t^2 + t^4 - \ldots)\,dt =$ result in text.
   (b) $\log(1 - x) = -\int_0^x dt/(1 - t) = -\int_0^x (1 - t)^{-1}\,dt = -\int_0^x (1 + t + t^2 + \ldots)\,dt$ $= -x - x^2/2 - x^3/3 - \ldots$.

(c) $\log \cos x = -\int_0^x \tan t \, dt$. Now use the given series for $\tan x$ to integrate term by term.

4. Let $x = 1$ in the answer to 3(a).

CHAPTER 20, SECTION 12

1. (a) Since $\sin x = x - x^3/3! + x^5/5! - \ldots$ we can express $\sin x/x$ as a series and integrate term by term to get the text's answer.
   (b) Since $\sin x^2 = x^2 - x^6/3! + x^{10}/5! - \ldots$, to evaluate $\int \sin x^2 \, dx$ we integrate term by term. Ans. $x^3/3 - x^7/7\cdot 3! + x^{11}/11\cdot 5! - \ldots + C$.
   (c) Using the series for $\sin x$ (see (a)) we have $\sqrt{x}\sin x = x^{3/2} - x^{7/2}/3! + x^{11/2}/5! - \ldots$. Hence $\int \sqrt{x}\sin x \, dx = 2x^{5/2}/5 - 2x^{9/2}/9\cdot 3! + 2x^{13/2}/13\cdot 5! - \ldots + C$.
2. (a) We know that $e^x = 1 + x + x^2/2! + \ldots$. Then $v \sim (32/k) - (32/k)\cdot(1 - kt + k^2t^2/2!) + \ldots$ or $v \sim 32t - 16kt^2$.
   (b) Yes, because in a vacuum $v = 32t$ for the motion described in (a).
3. (a) Use from (2a), $e^{-kt} = 1 - kt + k^2t^2/2! + k^3t^3/3! - \ldots$. Then substitute in the given expression for y and we get $y \sim 16t^2 - 16kt^3/3$.
   (b) Yes because as k approaches 0 we get $y = 16t^2$, which is the distance fallen in a vacuum.
4. (a) Use from (2a), $e^{-kt} = 1 - kt + k^2t^2/2! - \ldots$ and substitute this in the given expression for v. This gives the text's answer.
   (b) Yes because in the vacuum case for the motion described in (a), $v = v_0 - 32t$.
5. (a) Use the series for e^{-kt} and substitute in the given expression for y. This gives the text's result.
   (b) Yes because in the vacuum case, $y = -16t^2 + 1000t$.
6. (a) We know that $\log(1 + x) = x - x^2/2 + x^3/3 - \ldots$. Hence $\log(1 + 1000k/32) = 1000k/32 - (1000k/32)^2/2 + \ldots$. Multiply by $1/k$. This gives the text's approximate value for t_1.
   (b) Yes, the value for the vacuum case is $^{1000}/_{32}$.
7. (a) We know from Exercise 6(a) that $\log(1 + 1000k/32) = 1000k/32 - (1000k/32)^2/2 + (1000k/32)^3/3 - \ldots$. Use this approx. value to substitute in the expression for y_1. This gives the text's approximate answer.
   (b) Yes. The value of y_1 for the vacuum case is $(1000)^2/64$.
8. The series for arc tan x (see Ex. 3(a) of the previous list) is $x-x^3/3+x^5/5-\ldots$. Hence $\tan^{-1}\sqrt{k/32}\ 1000 = \sqrt{k/32}\ (1000) - (k/32)^{3/2}(1000)^{3/2}/3 + (k/32)^{5/2}(1000)^{5/2}/5 - \ldots$. Now substitute in the expression for t_1. This gives the text's approximate expression for t_1.

When k approaches 0 we get $t_1 = {}^{1000}/_{32}$ which is the value for the vacuum case.

9. (a) The series for $\log(1 + x)$ is given in Exercise 6(a). Let $x = 1000^2k/32$ and then substitute in the expression for y_1. This gives the text's answer.
   (b) Yes because y_1 in the vacuum case is $(1000)^2/64)$.
10. To three term $e^x = 1 + x + x^2/2$. Then $y = (c/2)(1 + x/c + x^2/2c^2 + 1 - x/c + x^2/2c^2) = c + x^2/2c$, which is the equation of a parabola.
11. We found in Exercise 10 the approximate value of y in terms x, namely, $y = c + x^2/2c$. Let $y = c + d$ and $x = \ell$. Then $c = \ell^2/2d$.
12. Use the value $W' = -(k/R^2)(1 - 2h/R)$. We have $W = (-k/R^2)\cdot(h - h^2/R) + C$. If we agree that $W = 0$ when $h = h_1$, say, we have that $C = (k/R^2)(h_1 - h_1^2/R)$. Then $W = -(k/R^2)\cdot(h - h^2/R - h_1 + h_1^2/R)$. Now let $h_1 = 500\cdot 5280$ and $h = 0$. Since $k = GMm$, $k = 32(4000\cdot 5280)^2\cdot(100)$. Then $W = 8{,}342\cdot 10^6$ ft.pdls. which is almost equal to the value $7{,}509\cdot 10^6$ ft.pdls. obtained in Chap. 6, Section 8, Exercise 2.
13. If we use two terms of the $\log(1 + x)$ series we have $t_2 = (1/k)[(kV \sin A/32) - (k^2V^2 \sin^2 A/2\cdot 32^2)]$. Since $T_2 = V \sin A/32$ we see that $t_2 < T_2$. Strictly one should add that if we assume that $kV \sin A/32$, which replaced x in the $\log(1 + x)$ series, is less than 1, then the terms decrease in size and then the error in neglecting all the remaining terms of the series is less than the value of the first term neglected. But the first term neglected is $k^3V^3 \sin^3 A/(3\cdot 32^3)$ and this is less than the term we subtract from T_2 to get T_1. Hence for $kV \sin A/32 < 1$, t_2 is rigorously shown to be less than T_2.
14. In this case if we carry three terms of the $\log(1 + x)$ series, we have, as in Exercise 13, that $y_2 < Y_2$. Again we can make a rigorous argument if $kV \sin A/32 < 1$.
15. If we use $e^x = 1 + x + x^2/2$, then $e^{-kt} = 1 - kt + k^2t^2/2$. We substitute this value of e^{-kt} in the expression for y, set $y = 0$ and solve for t. This gives $t_1 = 2V \sin A/(32 + kV \sin A)$. This is a useful result and does show that when $k = 0$ we get the time in a vacuum, namely, $V \sin A/16$. However if one wants to get the answer in terms of a series in k to the first power at least, one can expand by binomial theorem. Thus $t_1 = (2V \sin A/32)[1 + (kV \sin A/32)]^{-1} \sim (V \sin A/16)[1 - (kV \sin A/32)]$. In this form too we see that when $k = 0$, we get the result for a vacuum.
16. To get an answer to the first power in k we replace e^{-kt} by $1 - kt$ in the expression for x. Then replace t by the result in Exercise 15 and this gives the answer to this Exercise. Note that the range is less than in a a vacuum.

17. The problem is solved in the text.

Solutions to Chapter 21

CHAPTER 21, SECTION 2, FIRST SET

2. The perpendicular to the y-axis, for example, has the coordinates $(0, y, 0)$. Hence the three distances are $\sqrt{y^2 + z^2}$, $\sqrt{x^2 + z^2}$, and $\sqrt{x^2 + y^2}$.
3. (a) $(-3, 4, 5)$; (b) $(3, -4, 5)$; (c) $(3, 4, -5)$.
4. (a) yz-plane; (b) xz-plane; (c) xy-plane.
5. (a) z-axis; (b) y-axis; (c) x-axis.
6. (a) A plane parallel to and 7 units above the xy-plane.
   (b) A plane parallel to and 5 units to the left of the xz-plane.
7. (a) A cylinder whose axis is the z-axis and whose radius is 4 units.
   (b) A plane perpendicular to the xy-plane which intersects the xy-plane along the line $x = y$.

CHAPTER 21, SECTION 2, SECOND SET

1. Use distance formula. Ans. $3\sqrt{2}$.
2. 5
3. $\sqrt{45}$

CHAPTER 21, SECTION 2, THIRD SET

2. (a) The z-axis; (b) The y-axis
3. (a) Perpendicular to the z-axis; (b) perpendicular to the x-axis.
4. By equation (6) with $\alpha = \beta = \gamma$, we have $3\cos^2\alpha = 1$. Thus we find $\alpha = 54°44'$ approximately.
5. By equation (6) with $\alpha = 45°$, $\beta = 60°$, we have $\frac{1}{2} + \frac{1}{4} + \cos^2\gamma = 1$. Thus we have $\gamma = 60°$ or $\gamma = 180° - 60° = 120°$ as the two possible angles.
6. By equation (6) the sum of the squares of the direction cosines must equal unity. Applying this test we find that only (c) can be a set of direction cosines.
7. $-\frac{1}{2}, -\frac{1}{2}, -\sqrt{2}/2$.
8. Use (7) with AB given by (1). Then (b) $2/\sqrt{38}$, $3/\sqrt{38}$, $5/\sqrt{38}$; (d) $-1/\sqrt{3}$, $-1/\sqrt{3}$, $-1/\sqrt{3}$; (e) $-2/\sqrt{38}$, $-3/\sqrt{38}$, $-5/\sqrt{38}$.
9. The direction angles are $90°$, $0°$, $90°$ and the direction cosines are $0, 1, 0$.
10. By using (10) we get the direction cosines from the direction numbers. Hence (c) $\alpha = 80°16'$, $\beta = 59°32'$, $\gamma = 147°42'$.
11. We use (10) to get the direction cosines from the direction numbers. Hence
    (c) $\alpha = 14°3'$, $\beta = 75°57'$, $\gamma = 0$;
    (d) $\alpha = 122°19'$, $\beta = 143°18'$, $\gamma = 74°30'$;
    (e) $\alpha = 45°$, $\beta = 124°27'$, $\gamma = 64°54'$.

12. Yes, although the direction cosines are restricted by equation (6).
13. For any line we take a parallel line (or segment) from the origin to some point P with the direction from O to P that of the given directed line. Then the angle which the directed line OP makes with the positive x-axis is called the direction angle α and the angle which OP makes with the positive y-axis is called the direction angle β. As in the derivation of (5) we have $\cos^2\alpha + \cos^2\beta = 1$. The reason that we do not use two direction angles in plane analytic geometry is that one angle (the usual angle of inclination) suffices to fix the direction of a line, just as two angles really suffice in three-dimensions (except that in 3-dimensions we distinguish AB from BA).

CHAPTER 21, SECTION 2, FOURTH SET (p.)

1. (a) By using (10) to obtain $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ and then $\cos\alpha'$, $\cos\beta'$ and $\cos\gamma'$ we find that $\cos\alpha/\cos\alpha' = \cos\beta/\cos\beta' = \cos\gamma/\cos\gamma'$. Let r be the common ratio. Then $\cos\alpha = r\cos\alpha'$, $\cos\beta = r\cos\beta'$, $\cos\gamma = r\cos\gamma'$. Square and add these last three equations. Since the sum of the squares of the direction cosines is 1, we have $r = 1$ (or -1) and $\cos\alpha = \cos\alpha'$, $\cos\beta = \cos\beta'$, $\cos\gamma = \cos\gamma'$. Then the lines are parallel because they are parallel to the same line OP which defines their direction cosines.
   (b) Divide $aa' + bb' + cc' = 0$ by $\sqrt{a^2 + b^2 + c^2}\,\sqrt{a'^2 + b'^2 + c'^2}$. This gives, by (10), $\cos\alpha\,\cos\alpha' + \cos\beta\,\cos\beta' + \cos\gamma\,\cos\gamma' = 0$. By (15) the two lines are perpendicular.
2. Use the result of Exercise 1(b).
3. Let a, b, c be the direction numbers of the desired line. By the result of Exercise 1(b), $2a - 3b + 4c = 0$ and $-a + 2b + 3c = 0$. We have two equations in the three unknowns a, b and c. Solve for a and b in terms of c. Then $a = -17c$ and $b = -10c$. Take any value for c and we have three numbers which fix the desired a, b and c. For $c = -1$ we have the text answer.

CHAPTER 21, SECTION 3, FIRST SET

1. A: (3, 0, 0); B: (0, 3, 0); C: (0, 0, −6).
2. (b) A: (2, 0, 0); B: (0, −3, 0); C: (0, 0, 12).
   (d) A: (0, 0, 0); B: (0, 0, 0); C: (0, 0, 0).
   (e) A: (5, 0, 0); B: (0, 5, 0); C: None.
3. We use Theorem 2. This tells us what the direction numbers of the normal are. Then use (10). Hence (b) $2/\sqrt{6}, -1/\sqrt{6}, -1/\sqrt{6}$; (d) $2/\sqrt{10}$, 0, $-1/\sqrt{10}$; (e) 0, $1/\sqrt{2}$, $1/\sqrt{2}$.
4. Use Theorem 3. Then (b) $5/\sqrt{6}$; (c) $5/\sqrt{5}$; (d) $7/\sqrt{10}$; (e) $5/\sqrt{2}$.
5. (a) The direction numbers of the normal to the plane are 1, 1, 0. Hence the direction cosines of the normal are $1/\sqrt{2}$, $1/\sqrt{2}$, 0. The normal is perpendicular to the z-axis. Hence the plane itself is perpendicular to the xy-plane and so parallel to the z-axis.

(b) See (a). The plane is perpendicular to the yz-plane and so parallel to the x-axis.
(c) See (a). The plane is perpendicular to the xz-plane and so parallel to the y-axis.
(d) The direction numbers of the normal are 1, 0, 0 and these are also the direction cosines. Hence the normal is parallel to the x-axis. Hence the plane is perpendicular to the x-axis and so parallel to the yz-plane.
(e) See (d). The plane is perpendicular to the y-axis and so parallel to the xz-plane.
(f) See (d). The plane is perpendicular to the z-axis and so parallel to the xy-plane.

6. By Theorem 2, the direction numbers of the normals to the given planes are proportional. Hence (Exercise 1(a) of the preceding set), the normals are parallel and so the planes are parallel.

7. By Theorem 2 and Exercise 1(b) of the preceding set, the normals to the given planes are perpendicular and so the planes are.

8. (a) The line from (0, 0, 0) to (4, 5, 3) has direction numbers 4, 5, 3. Then the plane has the equation $4x + 5y + 3z + D = 0$. Since the plane passes through (1, 3, 2), $4 \cdot 1 + 5 \cdot 3 + 3 \cdot 2 + D = 0$ and so $D = -25$. Hence the text answer.
(b) Use the method in (a). Ans. $2x - 4y + 3z - 7 = 0$.

9. (a) The perpendicular distance from the origin to $x + y + z = 5$ is, by Theorem 3, $5/\sqrt{3}$. The perpendicular distance from the origin to $2x + 2y + 2z + 7 = 0$ is $7/\sqrt{12} = 7/2\sqrt{3}$. The difference of the distances is $\sqrt{3}/2$.
(b) Use the method in (a). Ans. $\sqrt{3}/2$.
(c) Use the method in (a). However the two planes are on opposite sides of the origin (see the footnote to Theorem 3) and so the two distances are added.

10. (a) Solve the three equations for the simultaneous solution.
(b) Same method as (a). Ans. $(3/4, 17/4, -3)$.

11. The desired equation can be written in the form $(A/D)x + (B/D)y + (C/D)z = 1$. This equation must be satisfied by the x,y and z of each of the given points. Hence we have 3 equations in the three unknowns, A/D, B/D and C/D. (The three points cannot lie on one line. If they do, there will not be a unique solution.)

12. This problem presupposes a knowledge of determinants. We note that by considering the determinant as expanded by minors of the first row, the given equation has the form of a plane $Ax + By + Cz + D = 0$. It remains to show that this plane contains the given points. Substituting $x = x_1$, $y = y_1$, $z = z_1$ into the determinent we see that it vanishes because two rows are identical. Thus (x_1, y_1, z_1) lies on the plane. Similarly (x_2, y_2, z_2) and (x_3, y_3, z_3) lie on the plane.

CHAPTER 21, SECTION 3, SECOND SET

1. Use (24) in each case. The answers to (b) and (d) are:
   (b) $78°45'$ or $101°15'$; (d) $72°32'$ or $117°28'$.

CHAPTER 21, SECTION 4, FIRST SET

1. To find the direction numbers use (31). To find the trace set $z = 0$ in the two equations of the planes and solve for x and y.
   (b) $a = -21$, $b = 0$, $c = 21$; trace: $(-10/3, -9, 0)$.
   (d) $a = -5$, $b = 1$, $c = -1$; no trace.
   (f) $a = 3$, $b = -5$, $c = -3$; trace: $(2, -2, 0)$.
   (g) $a = 3$, $b = 4$, $c = 4$; trace: $(1/4, -1/3, 0)$.
2. In general there are three sets of projecting planes, each consisting of any two of the three planes given below. Each equation is obtained by eliminating one letter from the two given equations.
   (a) $7y + 4z + 3 = 0$, $7x + 29z - 36 = 0$, $4x - 29y - 33 = 0$.
   (b) $y + 9 = 0$, $3x + 3z + 10 = 0$; the line lies in the plane $y = -9$ and so two of the projecting planes coincide.
   (c) $46y - 49z + 50 = 0$, $46x - 41z - 22 = 0$, $49x - 41y - 56 = 0$.
   (d) $y + z = 4$, $x - 3z = 8$, $x + 3y = 20$.
   (e) $8y = 9$, $4x = 23$; see (b).
   (f) $3x + z = 6$, $5x + y = 8$, $3y - 5z = -6$.
   (g) $4x - 3y = 2$, $4x - 3z = 1$, $3y - 3z = 1$.

CHAPTER 21, SECTION 4, SECOND SET

1. By definition the direction numbers are proportional to the direction cosines. Then by (36) the equations of the line may be written as $(x - x_1)/a = (y - y_1)/b = (z - z_1)/c$.
2. Yes, positive and negative values of d in (38) each yield half lines in opposite directions emanating from (x_1, y_1, z_1) with fixed direction angles α, β, γ.
3. Yes. To obtain this result, we note that any direction numbers may be written as $a = k\cos\alpha$, $b = k\cos\beta$, $c = k\cos\gamma$, where k is a constant of proportionality. Thus in place of (38) we obtain $x = x_1 + (d/k)a$, etc. Then replacing d by kt we obtain the desired result.
4. (a) By (36), $(x - 3)/(1/2) = (y - 4)/(1/\sqrt{2}) = (z - 5)/(-1/2)$ or by (38), $x = 3 + \frac{1}{2}d$, $y = 4 + (1/\sqrt{2})d$, $z = 5 - \frac{1}{2}d$.
   (b) $(x - 3)/(\sqrt{3}/2) = (y - 4)/(1/2) = (z + 4)/(1/\sqrt{2})$ or $x = 3 + (\sqrt{3}/2)d$, $y = 4 + \frac{1}{2}d$, $z = -4 + (1/\sqrt{2})d$.
   (c) A set of direction numbers is given by $1, -5, -3$. By Exercise 1, $(x - 3)/1 = (y + 2)/-5 = (z - 1)/-3$ or by Exercise 3, $x = 3 + t$, $y = -2 - 5t$, $z = 1 - 3t$.

(d) By (38), $x = 3$, $y = -2 - 2t$, $z = 1 + 4t$. Note that the representation given by (36) fails since it involves division by zero.

(e) By (38) and Exercise 3, $x = 3 + 2t$, $y = 2 + t$, $z = 1 + 3t$. Solving each equation for t and equating the results, we obtain the alternate representation $(x - 3)/2 = (y - 2)/1 = (z - 1)/3$.

(f) By Exercise 3, $x = t$, $y = -3 - 3t$, $z = 2 - t$ or $x/1 = (y + 3)/-3 = (z - 2)/-1$.

5. Parallel lines have the same direction numbers; hence a set of direction numbers of the desired line is $1, -3, -1$. Thus by Exercise 3, the answer is $x = t$, $y = -3 - 3t$, $z = 2 - t$.
6. Since parallel lines have the same direction numbers, a set of direction numbers of the desired line is $4, -3, -1$. (See Exercise 1.) Then by Exercise 3 the answer is $x = -2 + 4t$, $y = 4 - 3t$, $z = -t$.
7. Each value of d yields a point on the line.
8. Each value of t gives a point on the line. Calculate the coordinates of two points and draw the line joining them.
9. By Exercise 1, a set of direction numbers of the line is $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}$. By Theorem 2 of Section 3, a set of direction numbers of the normal to the plane is $2, 3, -3$. By Exercise 1(b) of Section 2, Third Set, we conclude that the line and normal are perpendicular from which the desired result follows.
10. Use the reasoning of Exercise 9 and Exercise 1(a) of Section 2, Third Set.
11. Verify that two points of the line lie in the plane. Thus if we let $z = 3$, then $y = -2$ and $x = 2$. Thus one point on the line is $(2, -2, 3)$. If we substitute these values in the equation of the plane they satisfy the equation.
12. (a) A set of direction numbers of the normal to the plane and hence of the line is $3, -1, 2$. Thus the line is $x = 2 + 3t$, $y = -3 - t$, $z = 4 + 2t$.

    (b) Use the method of (a). Hence $x = -1 + t$, $y = 3$, $z = 2 - 3t$.

    (c) Let the direction numbers of the line be a, b, c. The normals to the planes have respectively the direction numbers $1, 0, -2$ and $0, 1, -3$. By Exercise 1(b) of Section 2, Third Set, we thus have $a - 2c = 0$ and $b - 3c = 0$. Hence a set of direction numbers of the line is $2, 3, 1$ and the answer is $x = 2t$, $y = 2 + 3t$, $z = 4 + t$. Or use (31) to find the direction numbers of the line determined by the two planes.
13. Use equation (14) with the values $\cos\alpha = 3/\sqrt{14}$, $\cos\beta = 2/\sqrt{14}$, $\cos\gamma = 1/\sqrt{14}$ (the direction cosines of the line) and $\cos\alpha' = 3/\sqrt{14}$, $\cos\beta' = -1/\sqrt{14}$, $\cos\gamma' = 2/\sqrt{14}$ (the direction cosines of the normal to the plane) to obtain $\cos\theta = \frac{9}{14}$. Thus $\theta = 50°46'$.

CHAPTER 21, SECTION 5, FIRST SET

1. Use completing the squares as in the text to put the equation in the form (42).
   (b) $(2,3,0)$, $\sqrt{13}$; (d) $(0,6,-3)$, $5\sqrt{2}$; (e) $(2,-\frac{5}{2},-4)$, $\sqrt{5}/2$.
2. The point $(3,4,5)$. No other sets of (x,y,z) satisfy the equation because a sum of positive numbers cannot be 0.
3. Use equation (42).
   (b) $x^2+y^2+z^2-2y-4z-\frac{5}{4}=0$.
   (c) $x^2+y^2+z^2-6x+4z+12=0$.
   (d) $x^2+y^2+z^2+2x-6y-4z+11=0$.
4. (a) Upper portion of sphere of radius 5 and center $(2,3,0)$.
   (b) Upper portion of sphere of radius 5 and center $(2,3,3)$.
5. The radius of the sphere is the distance of $(2,-2,1)$ to the yz-plane. Hence the radius of the sphere is 2 and the equation is given by $x^2+y^2+z^2-4x+4y-2z+5=0$.
6. Let (x,y,z) be a point on the unknown surface.
   (a) The condition satisfied by (x,y,z) is $[(x-7)^2+(y-1)^2+(z+3)^2]^{1/2} = 2[(x+\frac{5}{4})^2+(y+2)^2+(z-\frac{3}{2})^2]^{1/2}$. Squaring, simplification and use of Theorem 2 leads to the desired result.
   (b) The condition satisfied by (x,y,z) is $[(x-4)^2+(y+5)^2+(z-1)^2] + [x^2+(y-2)^2+(z-4)^2]=64$. Simplification and the use of Theorem 2 gives the result.
7. Complete squares in the expression for the given sphere to find the common center, $(3,0,-2)$ of the two spheres. Thus the unknown sphere is given by $(x-3)^2+y^2+(z+2)^2=r^2$. Use the values $x=2$, $y=5$, $z=-7$ to determine that $r^2=51$.
8. It is geometrically obvious that the center of the sphere must lie on the perpendicular to the plane at the given point of tangency. Denoting the center by (x_0,y_0,z_0) and the given point by (x_1,y_1,z_1) we thus have that a set of direction numbers of the normal to the plane is given by $a=x_1-x_0$, $b=y_1-y_0$, $c=z_1-z_0$. Thus the plane is given by $ax+by+cz+d=0$. Since (x_1,y_1,z_1) lies on the plane, $d=-ax_1-by_1-cz_1$ and the plane is given by $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$. Use this method just described or the result.
   (a) $3x-2y+z-14=0$; (b) $2x-6y+3z+49=0$;
   (c) $2x+y+3z+18=0$.
9. Completing squares in (46) we obtain $[x+(G/2)]^2+[y+(H/2)]^2+[z+(I/2)]^2 = -K+[(G^2+H^2+I^2)/4]$. To have a real sphere the right side must be >0. Then if K is positive, $K<(G^2+H^2+I^2)/4$. If K is negative the right side is positive but then K still satisfies the condition just stated.

CHAPTER 21, SECTION 5, SECOND SET

1. The surfaces are all paraboloids.
   (a) Axis is z-axis; cross sections are circles for $z > 0$.
   (b) Axis is y-axis; cross sections are circles for $y > 0$.
   (c) Axis is x-axis; cross sections are circles for $x > 0$.
   (d) Axis is z-axis; cross sections are ellipses for $z > 0$ with x-axis as major axis.
   (e) Axis is x-axis; cross sections are ellipses for $x > 0$ with y-axis as major axis.
   (f) Axis is z-axis; cross sections are circles for $z > 5$. The surface lies above the plane. $z = 5$.
   (g) Axis is x-axis; cross sections are circles for $x > -4$. The surface lies above $x = -4$.
   (h) Axis is z-axis; cross section are ellipses for $z > 7$. Compare (d).
   (i) Axis is z-axis; cross sections are ellipses for $z > 0$. The semi-axes of the ellipses for $z = k$ are $a\sqrt{ck}$ and $b\sqrt{ck}$.
2. (a) In the yz-plane $x = 0$. Hence $z = 4y^2$ and $x = 0$ are the equations of the curve.
   (b) $z = x^2$, $y = 0$; (c) $x^2 + 4y^2 = 5$, $z = 5$; (d) $z = x^2 + 100$, $y = 5$.
3. (a) Since the paraboloid is a figure of revolution the parabola cut out by the xz-plane, in which $y = 0$, is a generating parabola. In the plane $y = 0$ the equation of the generating parabola is $z = (1/4p)\,x^2$. Thus the focus is $(0, 0, p)$.
   (b) Put $z = p$; then the circular section is $x^2 + y^2 = 4p^2$, $z = p$ and the diameter is thus $4p$.
4. (a) We must determine p. At $z = c$, we have $4pc = x^2 + y^2$. The radius of this circle is $2\sqrt{pc}$ and this should be a. Hence $4pc = a^2$ and $4p = a^2/c$. Then the equation of the paraboloid is $z = (c/a^2)(x^2 + y^2)$.
   (b) In exercise 3(a) we found that the coordinates of the focus of $z = (1/4p)(x^2 + y^2)$ are $(0, 0, p)$. Hence, in view of (a), the answer is $(0, 0, a^2/4c)$.

CHAPTER 21, SECTION 5, THIRD SET

1. The surfaces are all ellipsoids with centers at $(0, 0, 0)$.
   (a) $a = \sqrt{2}$, $b = 2$, $c = \sqrt{6}$;
   (b) $a = 6$, $b = 3$, $c = 2$;
   (c) $a = 10$, $b = 5$, $c = 10/3$;
   (d) $a = 10$, $b = c = 5$ (prolate spheroid);
   (e) $a = b = 3$, $c = 6$ (prolate spheroid).
2. The intersections are all ellipses.
   (a) $x^2 + 4y^2 = 36$, $z = 0$; $x^2 + 9z^2 = 36$, $y = 0$; $4y^2 + 9z^2 = 36$, $x = 0$.
   (b) $x^2 + 4y^2 = 36$, $z = 0$; $x^2 + 4z^2 = 36$, $y = 0$; $y^2 + z^2 = 9$, $x = 0$.

3. $x^2 + 4y^2 = 27$, $z = 1$.
4. Yes; the sphere corresponds to $a = b = c$ in equation (50).
5. The sections are given by $4y^2 + 9z^2 = 36 - k^2$. They are ellipses for $|k| < 6$, points for $|k| = 6$, and non-existent for $|k| > 6$.
6. Ellipsoid with center $(2, 3, 4)$ and with semi-axes $4, 5, 6$.

CHAPTER 21, SECTION 5, FOURTH SET

1. The surfaces are all hyperboloids of one sheet.
   (a) Axis is z-axis, cross section are ellipses with y-axis as major axis.
   (b) Axis is z-axis, cross sections are ellipses with x-axis as major axis.
   (c) Axis is z-axis, cross sections are ellipses with y-axis as major axis.
   (d) Axis is y-axis, cross sections are ellipses with z-axis as major axis.
   (e) Axis is y-axis, cross sections are ellipses with z-axis as major axis.
2. $4y^2 - z^2 = 36$, $x = 0$ (hyperbola); $9x^2 - z^2 = 36$, $y = 0$ (hyperbola); $9x^2 + 4y^2 = 36$, $z = 0$ (ellipse).
3. The curves are given by $4x^2 - z^2 = 36 - 9k^2$, $y = k$. For $|k| < 2$, they are hyperbolas with the z-axis as the major axis; for $|k| = 2$, they are intersecting lines; for $|k| > 2$, they are hyperbolas with the x-axis as the major axis.
4. Hyperboloid of revolution about the z-axis.
5. $x^2 + 2y^2 = 41$, $z = 3$.

CHAPTER 21, SECTION 5, FIFTH SET

1. $y^2/b^2 + z^2/c^2 = k^2/a^2 - 1$, $x = k$. These sections are non-existent for $k^2 < a^2$, points for $k^2 = a^2$, and the ellipses for $k^2 > a^2$.
2. The surfaces are all hyperboloids of two sheets.
   (a) Lies along the x-axis;
   (b) Lies along the x-axis;
   (c) Lies along the x-axis;
   (d) Lies along the x-axis;
   (e) Lies along the y-axis;
   (f) Lies along the z-axis.
3. $4x^2 - 2z^2 = 24 + 3k^2$, $y = k$. These sections are hyperbolas with the x-axis as the major axis.
4. $4x^2 - 9y^2 = 145$, $z = 4$.

CHAPTER 21, SECTION 5, SIXTH SET

1. $z = x^2/a^2$, $y = 0$. This is a parabola opening upward.
2. (a) Parabola opening downward with vertex $(0, 0, 0)$.
   (b) Parabola opening upward with vertex $(0, 5, -25/b^2)$.
   (c) Parabola opening downward with vertex $(k, 0, k^2/a^2)$.
   (d) Parabola opening upward with vertex $(0, k, -k^2/b^2)$.
3. The surfaces are all hyperbolic paraboloids with saddles at the origin.
   (a) Above the saddle the x-axis is the major axis of the cross section;
   (b) Above the saddle the y-axis is the major axis of the cross section;
   (c) This hyperbolic paraboloid "opens up" along the z-axis; that is the z-axis play the role of the x-axis in Fig. 21-23.
   (d) This hyperbolic paraboloid "opens up" along the y-axis; that is the y-axis plays the role of the x-axis in Fig. 21-23.
4. (a) Hyperbolic paraboloid of the type described in 3(d).
   (b) Hyperbolic paraboloid of the type described in 3(c).

CHAPTER 21, SECTION 5, SEVENTH SET

1. The surfaces are all cones with vertices at $(0, 0, 0)$.
   (a) Axis is z-axis; cross sections perpendicular to the z-axis are circles.
   (b) Axis is z-axis; cross sections perpendicular to the z-axis are ellipses.
   (c) Axis is x-axis; cross sections perpendicular to the x-axis are ellipses.
   (d) Axis is y-axis; cross sections perpendicular to the y-axis are ellipses.
2. The surface is generated by the lines emanating from the origin and intersecting the given circle. Lines emanating from the origin may be expressed as $x = at$, $y = bt$, $z = ct$, where t is a parameter. Suppose at $t = 1$ the lines intersect the circle; then $a^2 + b^2 = 4$, $c = 1$ and the line is given by $x = at$, $y = \pm\sqrt{4 - a^2}\,t$, $z = t$. Eliminating a and t, we find that all points (x, y, z) on the line satisfy the equation $x^2 + y^2 - 4z^2 = 0$.

CHAPTER 21, SECTION 5, EIGHTH SET

1. The surfaces are all cylinders.
   (a) Parallel to z-axis; cross sections are ellipses with centers $(0, 0, z)$.
   (b) Parallel to z-axis; cross sections are parabolas with foci $(1, 0, z)$.
   (c) Parallel to y-axis; cross sections are circles with centers $(0, y, 0)$.
   (d) Parallel to x-axis; cross sections are circles with centers $(x, 0, 0)$.
   (e) Parallel to y-axis; cross sections are parabolas with foci $(1, y, 0)$.
   (f) Parallel to z-axis; cross sections are circles with centers $(-2, 0, z)$.
   (g) Parallel to z-axis; cross sections are hyperbolas with asymptotes $x = \pm y$, $z = k$.
   (h) Parallel to z-axis; cross sections are hyperbolas with asymptotes $x = y = 0$, $z = k$.

Solutions to Chapter 22

CHAPTER 22, SECTION 1, FIRST SET

1. (b) 27; (d) -8; (f) -7.
2. We have $z = (x - y)^2$ which is always positive or 0.
3. (b) 0; (d) 1.
4. (b) 0; (d) 0.
5. (b) 1; (d) $^{253}/_{144}$.
6. $x^2 + y^2 \leq 1$.
7. (b) $6G/25$.

CHAPTER 22, SECTION 1, SECOND SET

1. The function $z = \sin(x - 3t)$ is inclined more to the x-axis. For example, when $x - 3t = \pi/2$ the points on the surface lie on a line which is the crest of the wave. But the line $x - 3t = \pi/2$ has a greater slope in the tx-plane.
2. The first function has an amplitude of 3.
3. The wave is inclined to the left whereas $\sin(x - ct)$ is inclined to the right. Or one says that the wave $z = \sin(x + ct)$ is moving to the left because when t increases x must decrease to keep $x + ct$ constant.
4. The points lie in a trough of the wave surface.

CHAPTER 22, SECTION 1, THIRD SET

1. As it stands, the function is undefined at $(0, 0)$ and so is not continuous there.
2. The function is not defined along $x = y$ and thus is not continuous there. If $x_0 = y_0$ and (x, y) approaches (x_0, y_0) through values such that $x > y$ then the function approaches positive infinity. If the approach is made through values such that $x < y$ then the function approaches negative infinity.
3. The surface consists of two half-planes both starting along the x-axis. One half-plane is inclined 45° to the positive y-axis and the other 135° to the positive y-axis. The function is differentiable except on the x-axis.
4. (a) Along the line $y = 2$ in the xy-plane.
   (b) Along the line $x = 2$ for $y \leq 2$ and along the line $y = 2$ for $x \leq 2$.

CHAPTER 22, SECTION 2

1. (b) $\partial V/\partial r = 2\pi rh$ means that the rate of change of volume with respect to the radius while the height is kept fixed, is the lateral surface area; $\partial V/\partial h = \pi r^2$ means that the rate of change of volume with respect to the height, while the radius is kept fixed is the cross-sectional area.
   (c) $\partial^2 V/\partial r\partial h = \partial(\pi r^2)/\partial r = 2\pi r$; $\partial^2 V/\partial h\partial r = \partial(2\pi rh)/\partial h = 2\pi r$.
2. (a) $z_y = -2a^2x^2y/(x^2+y^2)^2$.
   (b) $z_y = x\cos xy$.
   (c) $z_x = 2e^{2x}\sin y$; $z_y = e^{2x}\cos y$.
   (d) $z_y = 1/\sqrt{x^2+y^2}$.
3. $u_y = y/\sqrt{x^2+y^2}$.
4. $S_r = r(2r^2+h^2)/\sqrt{r^2+h^2}$; $S_h = \pi rh/\sqrt{r^2+h^2}$.
5. $z_x = x/(x^2+y^2)$, $z_y = y/(x^2+y^2)$, $z_{xx} = (-x^2+y^2)/(x^2+y^2)^2$, $z_{yy} = (-y^2+x^2)/(x^2+y^2)^2$. Hence $z_{xx}+z_{yy}=0$.
6. $z_x = \dfrac{x-1}{(x-1)^2+y^2} - \dfrac{(x+1)}{(x+1)^2+y^2}$, $z_y = \dfrac{y}{(x-1)^2+y^2} - \dfrac{y}{(x+1)^2+y^2}$,
   $z_{xx} = \dfrac{-(x-1)^2+y^2}{[(x-1)^2+y^2]^2} - \dfrac{-(x+1)^2+y^2}{[(x+1)^2+y^2]^2}$,
   $z_{yy} = \dfrac{(x-1)^2-y^2}{[(x-1)^2+y^2]^2} - \dfrac{(x+1)^2-y^2}{[(x+1)^2+y^2]^2}$.

   Hence $z_{xx}+z_{yy}=0$.
7. (a) $z_x = 2(x+ct)$, $z_t = 2c(x+ct)$, $z_{xx}=2$, $z_{tt}=2c^2$.
   (b) $z_x = e^{x+ct}$, $z_t = ce^{x+ct}$, $z_{xx} = e^{x+ct}$, $z_{tt} = c^2e^{x+ct}$.
   (c) See (a).
   (d) $z_x = \cos(x-ct)$, $z_t = -c\cos(x-ct)$, $z_{xx} = -\sin(x-ct)$, $z_{tt} = -c^2\sin(x-ct)$.
   (e) $z_x = -e^{x-ct}\sin(x-ct) + e^{x-ct}\cos(x-ct)$, $z_t = e^{x-ct}c\sin(x-ct) - ce^{x-ct}\cos(x-ct)$, $z_{xx} = -2e^{x-ct}\sin(x-ct)$, $z_{tt} = -2c^2e^{x-ct}\sin(x-ct)$.
8. $\partial z/\partial x = 2x$; at $x=2$, $\partial z/\partial x = 4$.
9. (a) $z_x = \dfrac{2x}{x^2+y^2}$, $z_y = \dfrac{2y}{x^2+y^2}$, $z_{xx} = \dfrac{-2x^2+2y^2}{(x^2+y^2)^2}$, $z_{yy} = \dfrac{2x^2-2y^2}{(x^2+y^2)^2}$.
   (b) $z_x = 2x$, $z_y = -2y$, $z_{xx}=2$, $z_{yy}=-2$.
   (c) $z_x = 6x$, $z_y = -6y$, $z_{xx}=6$, $z_{yy}=-6$.
   (d) $z_x = e^x\cos y$, $z_y = -e^x\sin y$, $z_{xx} = e^x\cos y$, $z_{yy} = -e^x\cos y$.
   (e) $z_x = -y/(x^2+y^2)$, $z_y = x/(x^2+y^2)$, $z_{xx} = 2xy/(x^2+y^2)^2$, $z_{yy} = -2xy/(x^2+y^2)^2$.
10. $u_x = 6x$, $u_{xy} = 0$.
11. $u_y = 4xy$, $u_{yx} = 4y$.

12. $u_y = e^{ax}\cos y \cos z$, $u_{yz} = -e^{ax}\cos y \sin z$.
13. $u_x = x/\sqrt{x^2 + y^2 + z^2}$.
15. $u_x = -x/(x^2 + y^2 + z^2)^{5/2}$. $u_{xx} = (2x^2 - y^2 - z^2)/(x^2 + y^2 + z^2)^{5/2}$. By symmetry we can write down u_{yy} and u_{zz}.

CHAPTER 22, SECTION 3, FIRST SET

1. The curve is $z = 2x^2 + 16$, $y = 4$. Then $\partial z/\partial x = 4x$ and at $x = 3$, $\partial z/\partial x = 12$. This is the slope of the curve.
2. The slope is given by $\partial z/\partial y$. At $(2,1,5)$, $\partial z/\partial y = 2$.
3. The slope is given by $\partial z/\partial x$. At $(1,1,4)$, $\partial z/\partial x = 3$. The slope is independent of x because the curve is a straight line.

4. (a) The equations of the curve are $x^2 + 3z^2 = 4$, $y = 2$. Then $2x + 6z\,(dz/dx) = 0$. and $dz/dx = -x/3z$. At $(1,2,1)$, $dz/dx = -1/3$.
   (b) The slope is 0 at $x = 0$. Since the equation of the curve (in the plane $y = 2$) is $x^2 + 3z^2 = 4$, $z = \pm\sqrt{4/3}$. The implicit equation defines two functions and so there are two answers $(0, 2, \sqrt{4/3})$ and $(0, 2, -\sqrt{4/3})$.
5. The surface is a right circular cone and the plane $y = 0$ cuts it in two lines whose equations are $z = x$, $y = 0$ and $z = -x$, $y = 0$. The slope on each line is constant.

CHAPTER 22, SECTION 3, SECOND SET

1. We use the theorem immediately above.
   (b) $8x - 12y - z - 10 = 0$; (d) $3x + 2y + 4z - 12 = 0$;
   (f) $\partial z/\partial x = 3\cos(x - 2y)$. At $x = \pi$, $y = \pi/4$, $\partial z/\partial x = 0$; $\partial z/\partial y = -6\cos(x - 2y)$. At $x = \pi$, $y = \pi/4$, $\partial z/\partial y = 0$. Hence by (19), $z - 3 = 0$ is the equation.
2. The normal to the sphere has the direction numbers $\partial z/\partial x$, $\partial z/\partial y$, -1 at the point $(2, 2, \sqrt{17})$. Hence since $z = \sqrt{25 - x^2 - y^2}$, the direction numbers are 2, 2, $\sqrt{17}$. To show that the normal lies along the radius we find the direction numbers of the radius. The radius joins $(2, 2, \sqrt{17})$ and $(0, 0, 0)$. Hence the direction numbers are the same as those of the normal and both emanate from the same point.
3. The normal at any point on the surface has the direction numbers $\partial z/\partial x$, $\partial z/\partial y$, -1 and these are $\sqrt{y/x}/2$, $\sqrt{x/y}/2$, -1. The direction numbers of the line joining (x, y, z) to $(0, 0, 0)$ are x, y, z. The two lines are perpendicular if $x\sqrt{y/x}/2 + y\sqrt{x/y}/2 - z(-1) = 0$. This is the case because $z = \sqrt{xy}$.

CHAPTER 22, SECTION 4

1. By (26), $dz/ds = f_x \cos\alpha + f_y \cos\beta = 2x\cos\alpha + 2y\cos\beta$. Since $x = 3$, $y = 4$, we have $dz/ds = 6\cos\alpha + 8\cos\beta$. We are told that $\alpha = 30°$ and since α is acute, the line is oriented either as in Fig. 22-12 or as in Fig. 22-13(c). In these two cases we readily find $\beta = 60°$, $\beta = 120°$ respectively. Thus $dz/ds = 3\sqrt{3} \pm 4$.
2. (a) Along $y = x + 1$ for increasing x, the situation is as in Fig. 22-12 where we may take P as (0, 1) and Q as (1, 2). For this line $\alpha = 45°$ and $\beta = 45°$. Then $\cos\alpha = \sqrt{2}/2$, $\cos\beta = \sqrt{2}/2$. By (26) at (3, 4), $dz/ds = 7\sqrt{2}$.

   (b) The situation is as in Fig. 22-13(b). Thus now α and β are supplements of their values in (a). Then $\cos\alpha = -\sqrt{2}/2$, $\cos\beta = -\sqrt{2}/2$, and, by (26), $dz/ds = -7\sqrt{2}$. This result must be the negative of the result in (a).
3. The situation is as in Fig. 22-12. Taking P as (0, 0) and Q as (1, 2), we find $\cos\alpha = 1/\sqrt{5}$, $\cos\beta = 2/\sqrt{5}$ and so, by (26), $dz/ds = 4\sqrt{5}/5$.
4. The situation is similar to that in Exercise 1. Here $\beta = 45°$ or $\beta = 135°$ and since $z_x = -2xy/(x^2 + y^2)^2$ and $z_y = (x^2 - y^2)/(x^2 + y^2)^2$, at (1, 2), $z_x = -4/25$ and $z_y = -3/25$. For $\alpha = 45°$ and $\beta = 45°$, $dz/ds = -7\sqrt{2}/50$. For $\alpha = 45°$, $\beta = 135°$, $dz/ds = -\sqrt{2}/50$.
5. (a) According to the result obtained from (25) the direction numbers of the direction of maximum rate of change are f_x, f_y, f_z. Thus the direction cosines of this direction are $f_x/\sqrt{f_x^2 + f_y^2 + f_z^2}$, etc. Thus in two dimensions the angles made with the x- and y-axes respectively are $\cos\alpha = f_x/\sqrt{f_x^2 + f_y^2}$, $\cos\beta = f_y/\sqrt{f_x^2 + f_y^2}$. Here we find $\cos\alpha = 8/\sqrt{145}$, $\cos\beta = 9/\sqrt{145}$.

   (b) The normal to any curve is the negative reciprocal of its slope. Using implicit differentiation, we find $8x + 18yy' = 0$. Thus at the point in question, $y' = -8/9$. Thus the slope of the normal at this point is 9/8 and so $\tan\alpha = 9/8$. It is easy to see that this agrees with the direction obtained in (a).

   (c) According to (a), the maximum rate of change of $z = 4x^2 + 9y^2$ at the point (2, 1) takes place in the direction $\cos\alpha = 8/\sqrt{145}$ and $\cos\beta = 9/\sqrt{145}$. There this maximum is $dz/ds = f_x\cos\alpha + f_y\cos\beta = 16(8/\sqrt{145}) + 18(9/\sqrt{145}) = 290/\sqrt{145} = \sqrt{145}$. If we calculate $(z_x^2 + z_y^2)^{1/2}$ for the function $z = 4x^2 + 9y^2$ at the point (2, 1) we obtain the same result.
6. (a) This is just a special case of (28) and the conclusions deduced from it. We have but to put f_z and $\cos\gamma = 0$.

   (b) See (a).

7. A set of direction numbers in the desired direction is $7-(-1)$, $7-1$, $7-7$ or $8, 6, 0$. Thus $\cos\alpha = 8/\sqrt{100}$, $\cos\beta = 6/\sqrt{100}$, $\cos\gamma = 0$. By use of (27) we find $du/ds = 10$ in this direction at the point $(-1, 1, 7)$.
8. At the given point $(2, 4)$ we find that $z_x = -3$ and $z_y = 1$. To find the α and β, note that the slope of $y = x^2$ at the point $(2, 4)$ is 4. Then $\tan\alpha = 4$ and so $\cos\alpha = 1/\sqrt{17}$. For the direction of increasing x, β is the complement of α so that $\cos\beta = 4/\sqrt{17}$. Then $f_x\cos\alpha + f_y\cos\beta = 1/\sqrt{17}$.
9. According to the conclusion drawn from (28), the direction numbers of the maximum directional derivative are f_x, f_y and f_z. In our case $f_x = y + z$, $f_y = x + z$, $f_z = y + x$. At $(-1, 1, 7)$, $f_x = 8$, $f_y = 6$, $f_z = 0$. This is the required direction. The value of the maximum directional derivative is $\sqrt{f_x^2 + f_y^2 + f_z^2} = 10$.
10. We use (27). Here $f_x = x/(x^2 + y^2 + z^2)$, $f_y = y/(x^2 + y^2 + z^2)$ and $f_z = z/(x^2 + y^2 + z^2)$. At $(3, 4, 8)$, $f_x = 3/89$, $f_y = 4/89$ and $f_z = 8/89$. The direction from $(3, 4, 8)$ to the point $(5, 7, 10)$ has the direction numbers $2, 3, 2$ and the direction cosines $2/\sqrt{17}$, $3/\sqrt{17}$, $2/\sqrt{17}$. Then, by (27), $du/ds = 34/89\sqrt{17}$.
11. We use (27). Here at $(1, -1, -1)$, $f_x = 2$, $f_y = -4$, $f_z = -2$. The direction from $(1, -1, -1)$ to $(3, -2, 5)$ has the direction numbers $2, -1, 6$ or the direction cosines $2/\sqrt{41}$, $-1/\sqrt{41}$, $6/\sqrt{41}$. Then by (27), $du/ds = -4/\sqrt{41}$.

CHAPTER 22, SECTION 5

1. By use of (38) and (39) we find $u = s^2t^2 + 5s^2 + 8st + 5t^2$. Thus $\partial u/\partial s = 2st^2 + 10s + 8t$ in agreement with (47).
2. $\partial u/\partial t = u_x\,\partial x/\partial t + u_y\,\partial y/\partial t + u_z\,\partial z/\partial t = 2x\cdot s + 2y\cdot 2 + 2z\cdot 1 = 2xs + 4y + 2z$. Replacing x, y, and z by their values in terms of s and t we obtain $\partial u/\partial t = 2s^2t + 8s + 10t$.
3. (a) This is a special case of (30) where now x and y are functions of t only. Hence, by (32), $dz/dt = z_x\,dx/dt + z_y\,dy/dt$.
   (b) This is a special case of (a) where x is independent of t. Hence $dx/dt = 0$ and $dz/dt = 3y\,dy/dt$.
   (c) This is a special case of (a). Apply the result of (a) and then since $x = t$, $dx/dt = 1$ and $dy/dt = dy/dx$.
4. (a) Following the suggestion we write $z = f(u)$, $u = x + ct$. To find z_x we can apply the ordinary chain rule because t is regarded as a constant. Then $z_x = (\partial f/\partial u)\cdot 1$ or f_u.
   (b) The argument is the same as in (a). Then $z_t = (\partial f/\partial u)(\partial u/\partial t) = c\,\partial f/\partial u$.
   (c) $z_{xx} = \partial(f_u)/\partial x$. Now f_u is a function of u and u is a function of x. Hence we apply the ordinary chain rule to f_u. The result is $f_{uu}\cdot 1$ or f_{uu}. Like-

wise to find z_{tt} we want $\partial(z_t)/\partial t$. Since $z_t = c\, f_u$ and f_u is a function of u and u is a function of t we apply the ordinary chain rule to $c\, f_u$. Then $z_{tt} = c\, f_{uu}(\partial u/\partial t) = c\, f_{uu}\, c = c^2 f_{uu}$. Hence $z_{tt} = c^2 z_{xx}$.

5. Let $x - y = u$ and $y - x = v$. Then $z = f(u, v)$ where $u = x - y$ and $v = y - x$. Now apply (31) to obtain z_x. $z_x = f_u\, \partial u/\partial x + f_v\, \partial u/\partial x = f_u \cdot 1 + f_v(-1)$. Use (32) to obtain z_y. Then $z_y = f_u\, \partial u/\partial y + f_v\, \partial v/\partial y = f_u(-1) + f_v \cdot 1$. Hence $z_x + z_y = 0$.
6. In finding u_x we may use the ordinary chain rule because y is regarded as a constant. Then $u_x = (du/dr)(\partial r/\partial x) = (du/dr)(x/r)$. Now to find u_{xx} we must differentiate once more with respect to x. u_x is a product of two functions of x. Hence $u_{xx} = (x/r)\,\partial(du/dr)/\partial x + (du/dr)\,\partial(x/r)/\partial x$. Since du/dr is a function of r and r is a function of x we must use the ordinary chain rule again. Then $u_{xx} = (x/r)(d^2u/dr^2)(\partial r/\partial x) + (du/dr)[r - (\partial r/\partial x)]/r^2$ $= (x/r)(d^2u/dr^2)(x/r) + (du/dr)\cdot[r - (x/r)]\, r^2 = (x^2/r^2)(d^2u/dr^2) + (du/dr)$ (y^2/r^3). Likewise $u_{yy} = (y^2/r^2)(d^2u/dr^2) + (du/dr)(x^2/r^3)$. If we add u_{xx} and u_{yy} we obtain $d^2u/dr^2 + (1/r)(du/dr)$, because $x^2 + y^2 = r^2$.
7. We have $z = f(x, y)$ with $x = \rho\cos\theta$ and $y = \rho\sin\theta$. We use (31). Then $z_\rho = f_x(\partial x/\partial\rho) + f_y(\partial y/\partial\rho) = f_x\cos\theta + f_y\sin\theta$. By (32), $z_\theta = f_x(\partial x/\partial\theta) + f_y(\partial y/\partial\theta)$ $= f_x(-\rho\sin\theta) + f_y(\rho\cos\theta)$. Then $z_\rho^2 + (1/\rho^2)\, z_\theta^2 = z_x^2 + z_y^2$. Alternatively we can start with $z = F(\rho, \theta)$, $\rho = \sqrt{x^2 + y^2}$, $\theta = \arctan(y/x)$ and use (31) and (32).
8. One can repeat the steps from (44) to (50) except that y replaces x and, of course, we use the proper values for $\partial\rho/\partial y$, $\partial\theta/\partial y$, $\partial^2\rho/\partial y^2$ and $\partial^2\rho/\partial\theta^2$.
9. We start with $z = f(x, y)$, $x = \rho\cos\theta$, $y = \rho\sin\theta$. Then, by (32), $z_\theta = f_x(\partial x/\partial\theta)$ $+ f_y(\partial y/\partial\theta) = f_x(-\rho\sin\theta) + f_y(\rho\cos\theta)$. Now to differentiate once again with respect to θ we must recognize that each term is a product. To differentiate f_x with respect to θ we must use (32) again because f_x is a function of x and y and the latter are functions of θ. Likewise to differentiate f_y with respect to θ. Hence $z_{\theta\theta} = (\partial(f_x)/\partial\theta)(-\rho\sin\theta) + f_x(-\rho\cos\theta) + (\partial(f_y)/\partial\theta)\cdot$ $(\rho\cos\theta) + f_y(-\rho\sin\theta) = [f_{xx}(\partial x/\partial\theta) + f_{xy}\partial y/\partial\theta](-\rho\sin\theta) - f_x\rho\cos\theta + [f_{yx}(\partial x/\partial\theta)$ $+ f_{yy}(\partial y/\partial\theta]\rho\cos\theta - f_y\rho\sin\theta$. If we now put in the values of $\partial x/\partial\theta$ and $\partial y/\partial\theta$ we obtain the text's answer.
10. We first repeat the steps of Exercise 9 replacing θ by ρ and of course the proper values of $\partial x/\partial\rho$ and $\partial y/\partial\rho$. In the course of this work we also obtain z_ρ, namely, $z_\rho = f_x(\partial x/\partial\rho) + f_y(\partial y/\partial\rho)$. With $z_{\theta\theta}$ from Exercise 9 and the values of z_ρ and $z_{\rho\rho}$ we form the right side of (52) and this equals the left side. The calculations yield $z_\rho = f_x\cos\theta + f_y\sin\theta$. $z_{\rho\rho} = f_{xx}\cos^2\theta$ $+ 2f_{xy}\sin\theta\cos\theta + f_{yy}\sin^2\theta$.
11. (a) By (31), $u = u_x(\partial x/\partial\rho) + u_y(\partial y/\partial\rho)$. Since $x = \rho\cos\theta$ and $y = \rho\sin\theta$, $u_\rho = u_x\cos\theta + u_y\sin\theta$. We now calculate v_θ. By (32), $v_\theta = v_x(\partial x/\partial\theta)$ $+ v_y(\partial y/\partial\theta) = v_x(-\rho\sin\theta) + v_y(\rho\cos\theta)$. If we now use the fact that $u_x = v_y$ and $u_y = -v_x$ we see that $u_\rho = (1/\rho)v_\theta$.
    (b) The method is exactly the same as in (a).
12. This is merely an extension of (33) and (34). No proof is required.

13. To make the situation come under a known chain rule we can write $x = s$, $y = t$. Then $u = f(x, y, z)$ where $x = s$, $y = t$, $z = k(s, t)$. We may now apply (40), and since $\partial y/\partial s = 0$ and $\partial x/\partial s = 1$ we get the text answer.
14. The problem is essentially the same as that worked out in the text where $z_{xx} + z_{yy}$ is transformed to polar coordinates. The only difference here is that instead $x = \rho\cos\theta$ and $y = \rho\sin\theta$ we have $x = r(e^{\theta} + e^{-\theta})/2$, $y = r(e^{\theta} - e^{-\theta})/2$. This difference means that the quantities $\partial\rho/\partial x$, $\partial\theta/\partial x$, etc. are different. If we work from $F(r, \theta)$, as the text does, we need r and θ as function of x and y. These are more complicated than x and y as functions of r and θ. Hence start with $z = f(x, y)$ and compute z_r, z_{rr} and $z_{\theta\theta}$. Then show that $z_{rr} - (1/r^2)z_{\theta\theta} + (1/r)z_r = z_{xx} - z_{yy}$.

CHAPTER 22, SECTION 6

1. $3z^2(\partial z/\partial y) - 6z(\partial z/\partial y) + 2x = 0$. Solve for $\partial z/\partial y$.
2. $y + y(\partial z/\partial x) - 2z(\partial z/\partial x) = 0$. Hence $\partial z/\partial x = y/(2z - y)$. $x + y(\partial z/\partial y) + z - 2z(\partial z/\partial y) = 0$. Hence $\partial z/\partial y = (x + z)/(2z - y)$.
3. $$\partial^2 z/\partial x^2 = \frac{(3z^2 - 6z)(-2) - (-2x - 2y)[6z(\partial z/\partial x) - 6(\partial z/\partial x)]}{(3z^2 - 6z)^2}.$$

   Now simplify and replace $\partial z/\partial x$ by its value. The answer is in the text.
4. (a) To find z_y differentiate thus: $2y + 2z(\partial z/\partial y) - 4x(\partial z/\partial y) = 0$. Then $\partial z/\partial y = y/(2x - z)$

   (b) To find z_x: $3x^2 + 3z^2(\partial z/\partial x) - 4xy(\partial z/\partial x) - 4yz = 0$. Then $z_x = (3x^2 - 4yz)/(4xy - 3z^2)$. To find z_y the process is exactly the same. $z_y = (3y^2 - 4xz)/(4xy - 3z^2)$

   (c) To differentiate with respect to x we must think in terms of $\sin u$ with $u = x + y + z$ and z as a function of x and y. Then $(\cos u)(1 + \partial z/\partial x) = 0$. Then $z_x = -1$. Likewise to find z_y: $(\cos u)(1 + \partial z/\partial y) = 0$ and $z_y = -1$.

   (d) To find z_x: $3z^2(\partial z/\partial x) + 3x(\partial z/\partial x) + 3z = 0$. Then $z_x = -z/(x + z^2)$. Likewise to find z_y: $3z^2(\partial z/\partial y) + 3x(\partial z/\partial y) + 4 = 0$. Then $z_y = -4/(3x + 3z^2)$.
5. $3z^2(\partial z/\partial x) + 3x(\partial z/\partial x) + 3z = 0$. Then $z_x = -z/(z^2 + x)$. $3z^2(\partial z/\partial y) + 3x(\partial z/\partial y) - 3 = 0$. Then $z_y = -1/(z^2 + x)$. $z_{xx} = 2zx/(z^2 + x)^3$; $z_{yy} = -2z/(z^2 + x)^3$. Hence the result.
6. This exercise involves just a change of letters in (57) and (58): $\partial x/\partial y = -F_y/F_x$; $\partial x/\partial z = -F_z/F_x$.
7. By (61) the normal to the surface $F(x, y, z) = 0$ or to the tangent plane of the surface at a point (x_0, y_0, z_0) has the direction numbers $F_x(x_0, y_0, z_0)$, $F_y(x_0, y_0, z_0)$, $F_z(x_0, y_0, z_0)$. Since (x_0, y_0, z_0) lies on the tangent plane, the tangent plane is given by $F_x \cdot (x - x_0) + F_y \cdot (y - y_0) + F_z \cdot (z - z_0) = 0$ where

F_x, F_y, F_z are evaluated at $x = x_0$, $y = y_0$, $z = z_0$.

(a) Take $F(x, y, z) = x^2 + y^2 + z^2 - 25 = 0$. To obtain the tangent plane we have: $x_0x + y_0y + z_0z = x_0^2 + y_0^2 + z_0^2$ or $x_0x + y_0y + z_0z = 25$.

(b) $16(x_0 - 3)(x - x_0) + 25(y_0 - 2)(y - y_0) + 400(z_0 - 1)(z - z_0) = 0$.

(c) $3x_0x + 4y_0y - 5z_0z = 3x_0^2 + 4y_0^2 - 5z_0^2 = 10$.

(d) $(y_0 + 3)(x - x_0) + (x_0 + z_0)(y - y_0) + y_0(z - z_0) = 0$.

(e) $x_0x + z_0z = x_0^2 + z_0^2 = 1$.

8. To show that the tangent planes at the common point are perpendicular we can show that the normals to the planes are perpendicular. We use (61). For the ellipsoid at the given point the direction numbers of the normal are $24\sqrt{5}/5$, $8\sqrt{2}$, $32\sqrt{5}/5$. For the hyperboloid the direction numbers of the normal are $8\sqrt{5}/5$, $4\sqrt{2}$, $-16\sqrt{5}/5$. To show that the two normals are perpendicular we use the condition $aa' + bb' + cc' = 0$ which was established in Chapter 21. Here we have $(24\sqrt{5}/5)(8\sqrt{5}/5) + (8\sqrt{2})(4\sqrt{2}) + (32\sqrt{5}/5)(-16\sqrt{5}/5)$ and this is 0.

9. Direction numbers for the normal to the first surface at (x_0, y_0, z_0) are $3x_0$, $4y_0$, $8z_0$ and for the second surface are x_0, $2y_0$, $-4z_0$. The condition for orthogonality of the normals and hence of the surfaces is $3x_0 \cdot x_0 + 4y_0 \cdot 2y_0 + 8z_0 \cdot (-4z_0) = 3x_0^2 + 8y_0^2 - 32z_0^2 = 0$. Since (x_0, y_0, z_0) is a common point of the surfaces we have $3x_0^2 + 4y_0^2 + 8z_0^2 = 24$ and $x_0^2 + 2y_0^2 - 4z_0^2 = 4$. Solving these two equations for x_0^2 and y_0^2 in terms of z_0^2 gives $x_0^2 \mp 16 - 16z_0^2$, $y_0^2 = 10z_0^2 - 6$. If we substitute these values of x_0^2 and y_0^2 in the orthogonality condition we see that it is satisfied.

10. The method is precisely the same as in Exercise 8. The direction numbers of the normal to the ellipsoid at the given point are $24\sqrt{5}/5$, $8\sqrt{2}$, $32\sqrt{5}/5$. The direction numbers of the normal to the hyperboloid are $32\sqrt{5}/5$, $-8\sqrt{2}$, $-4\sqrt{5}/5$. The expression $(24\sqrt{5}/5)(32\sqrt{5}/5) - (8\sqrt{2})(8\sqrt{2}) + (32\sqrt{5}/5)(-4\sqrt{5}/5) = 0$.

11. The method is precisely the same as in 9. At (x_0, y_0, z_0) the normal to the ellipsoid has direction numbers $6x_0$, $8y_0$, $16z_0$ and the normal to the hyperboloid has direction numbers $8x_0$, $-8y_0$ $-2z_0$. We must show that $6x_0 \cdot 8x_0 + 8y_0(-8y_0) + 16z_0(-2z_0) = 0$. If (x_0, y_0, z_0) is common to the two surfaces it satisfies both equations. Solve these for x_0^2 and y_0^2 in terms of z_0^2. The solutions are $x_0^2 = 4 - z_0^2$, $4y_0^2 = 12 - 5z_0^2$. If we substitute these values in the condition for perpendicularity we see that it is 0.

12. The surface $u = \text{const.}$ is $f(x, y, z) = 0$. By (61) the normal to this surface at say (x_0, y_0, z_0) has the direction of the maximum du/ds.

13. We know by (60) that $dy/dx = -F_x/F_y$. Likewise by (60) if we regard x as a function of y, $dx/dy = -F_y/F_x$.

(c) $f_x = 4x^3 + 32$, $f_y = 4y^3 - 4$, $f_{xx} = 12x^2$, $f_{yy} = 12y^2$, $f_{xy} = 0$. We solve simultaneously $x^3 + 8 = 0$, $y^3 - 1 = 0$. Hence $(-2, 1)$ is a possible minimum or maximum point. At $(-2, 1)$, $f_{xy}^2 - f_{xx}f_{yy} < 0$ and $f_{xx} > 0$. Hence $(-2, 1)$ is a minimum point and the minimum is 1.

3. We wish to minimize the distance of a point on the plane from the origin. Equivalently we minimize $u = x^2 + y^2 + z^2$ subject to the condition $3x + 4y - z = 26$. Considering z as a function of x and y, we find $u_x = 2x + 2zz_x$, $u_y = 2y + 2zz_y$. Since $z = 3x + 4y - 26$, we have $z_x = 3$, $z_y = 4$. Thus the conditions $u_x = 0$, $u_y = 0$ yield $2x + 6z = 0$, $2y + 8z = 0$ for the minimum. From these we have $x = -3z$ and $y = -4z$. If we substitute these values in the equation of the plane we obtain $z = -1$ and we find that the minimum occurs at $(3, 4, -1)$. We could use $u_x = 2x + 6z$, $u_y = 2y + 8z$ to apply (b) and (c) or (b) and (c′) of the theorem.

4. From (67), $z = (12 - xy)/(x + y)$ and so (66) becomes $V = (12xy - x^2y^2)/(x + y)$. Equating V_x and V_y to zero, we obtain the relations $x^2 + 2xy = 12$, $y^2 + 2xy = 12$ (after excluding the values $x = 0$, $y = 0$). By subtracting the second equation from the first it then follows that $x = 2$, $y = 2$ and by the relation for z that $z = 2$.

5. If the box has no top then (67) is replaced by the relation $xy + 2yz + 2zx = 24$. Using either the method of the text or the method of Exercise 4, we find that the maximum volume occurs for $x = y = 2\sqrt{2}$, $z = \sqrt{2}$.

6. Let the angles be denoted by A, B, C. Then we are to maximize $P = \sin A \cdot \sin B \sin C$. Since $A + B + C = \pi$, we have $P = \sin A \sin B \sin(\pi - A - B) = \sin A \sin B \sin(A + B)$. The relations $dP/dA = 0$, $dP/dB = 0$ yield $\cos A \cdot \sin(A + B) + \sin A \cos(A + B) = 0$, $\cos B \sin(A + B) + \sin B \cos(A + B) = 0$. Multiplying the first of these by $\cos B$ and the second by $\cos A$ and subtracting leads to $\sin(A - B) = 0$ or $A = B$. Putting $B = A$ in either of these relations leads to $\tan A = \tan B = \sqrt{3}$ or $A = B = 60°$. Now $C = 180° - 120° = 60°$ also and so the triangle is equiangular. By elementary geometry this implies that the triangle is also equilateral.

7. We are to minimize $S = 2(xy + yz + zx)$ with the restriction that $xyz = V$ = given constant. This is equivalent to minimizing $S = 2xy + 2(x + y)(V/xy)$ with respect to x and y. We find in the usual way that $x = y = \sqrt[3]{V}$ and so $z = V/xy = \sqrt[3]{V}$ and the box is a cube.

8. We have $V = 4\pi abc/3$. Let $\ell = a + b + c$. Then $c = \ell - a - b$, and $V = 4\pi ab(\ell - a - b)/3 = (4\pi ab\ell - 4\pi a^2b - 4\pi ab^2)/3$. Then $V_a = (4\pi b\ell - 8\pi ab - 4\pi b^2)/3$ and $V_b = (4\pi a\ell - 4\pi a^2 - 8\pi ab)/3$. Setting $V_a = 0$ and $V_b = 0$ we find that $a = \ell/3$ and $b = \ell/3$; then $c = \ell/3$. Hence a sphere.

9. We find $z_x = y$, $z_y = x$. Thus the only candidate for a max. or min. is $x = 0$, $y = 0$. However $z_{xy}^2 - z_{xx}z_{yy} = 1 > 0$ and so there is no max. or min.

10. We are to maximize $P = xyz$ under the condition that $x + y + z = 12$. Then $z = 12 - x - y$ and $P = xy(12 - x - y) = 12xy - x^2y - xy^2$. Now find P_x and P_y and set both equal to zero. We have $12y - 2xy - y^2 = 0$ and $12x - x^2 - 2xy = 0$. Divide through by y and x, respectively, and we obtain $x = y = 4$. Then $z = 4$.

11. (a) Instead of minimizing the distance let us minimize the square of the distance (when one is a minimum the other is). Hence we wish to minimize $F = x^2 + y^2 + z^2$ with $x^2 - z^2 = 1$. If we use $x^2 = 1 + z^2$ we obtain $F = y^2 + 2z^2 + 1$. Then $F_y = 2y$ and $F_z = 4z$. Hence $y = 0$ and $z = 0$ and from $x^2 - z^2 = 1$, $x = \pm 1$. Thus $(1, 0, 0)$ and $(-1, 0, 0)$ furnish minima.

    Note that if we eliminate z from F we obtain $F = 2x^2 + y^2 - 1$. Then $F_x = 0$ and $F_y = 0$ yield $x = 0$ and $y = 0$ and, from $x^2 - z^2 = 1$, $z = \pm\sqrt{-1}$, which is no real solution. The difficulty is that $x = 0$ does not belong to any point on the surface $x^2 - z^2 = 1$. We can see from F itself that it does have a minimum at $x = 0$, $y = 0$ because for any other x and y, $F > -1$.

    (b) The direction numbers of the normal to the surface $f(x, y, z) = x^2 - z^2 - 1 = 0$ are $2x, 0, -2z$ and at $(\pm 1, 0, 0)$ these direction numbers are $\pm 2, 0, 0$. The direction numbers of the line joining $(\pm 1, 0, 0)$ to the origin $(0, 0, 0)$ are $(\pm 1, 0, 0)$. This line is parallel to the normal and so must itself be the normal.

12. We first find conditions on (x, y, z) so that the distance will be least. Thus we minimize $F = (x - a)^2 + (y - b)^2 + (z - c)^2$ subject to $\phi(x, y, z) = 0$. Considering z as a function of x, y, we set $F_x = 0$, $F_y = 0$ and obtain $(x - a) + (z - c)z_x = 0$, $(y - b) + (z - c)z_y = 0$. From $\phi = 0$, we conclude that $z_x = -\phi_x/\phi_z$, $z_y = -\phi_y/\phi_z$ and thus the conditions for a minimum become $(x - a)\phi_z = (z - c)\phi_x$ and $(y - b)\phi_z = (z - c)\phi_y$. At any point (x_0, y_0, z_0) where these conditions are satisfied we have $\phi_x(x_0, y_0, z_0) = \lambda(x_0 - a)$, $\phi_y(x_0, y_0, z_0) = \lambda(y_0 - z)$ and $\phi_z(x_0, y_0, z_0) = \lambda(z_0 - c)$ where λ is a constant of proportionality. Now the normal to $\phi = 0$ at a point (x_0, y_0, z_0) has direction numbers $\phi_x(x_0, y_0, z_0)$, $\phi_y(x_0, y_0, z_0)$, $\phi_z(x_0, y_0, z_0)$. The equations of the normal are $x = x_0 + \phi_x(x_0, y_0, z_0)d$, $y = y_0 + \phi_y(x_0, y_0, z_0)d$, $z = z_0 + \phi_z(x_0, y_0, z_0)d$, where d is the parameter. Using the relations derived above for ϕ_x, ϕ_y, ϕ_z at the point (x_0, y_0, z_0), the equations take the form $x = x_0 + (x_0 - a)t$, $y = y_0 + (y_0 - b)t$, $z = z_0 + (z_0 - c)t$ where $t = \lambda d$ is the parameter. Since this line passes through (a, b, c)(for $t = -1$), the derived conclusion follows.

CHAPTER 22, SECTION 9

1. Here $F(x, y, \alpha) = y - (x - \alpha)^3$, so that the equations (87) become $y = (x - \alpha)^3$ and $(x - \alpha)^2 = 0$. Thus $\alpha = x$ and so the envelope is given by $y = 0$. A sketch shows that in this case the envelope is just the locus of the inflection points of the family of curves $y = (x - \alpha)^3$.

2. Here the condition $F_a = 0$ yields $x = a$ and so the envelope is $y = \pm 1$. Note that since the family of curves is just the family of circles with centers on the x-axis and radius 1, the conclusion is geometrically obvious.
3. (a) The conditions (79) yield $2\alpha y = 2x + \alpha^2$, $\alpha = y$. Thus the envelope is $2y^2 = 2x + y^2$ or $y^2 = 2x$.
   (b) The conditions (79) yield $x\cos\alpha + y\sin\alpha = 2$, $-x\sin\alpha + y\cos\alpha = 0$. By solving simultaneously for x and y we find $x = 2\sin\alpha$, $y = 2\cos\alpha$ and by squaring and adding the envelope is $x^2 + y^2 = 4$.
   (c) If we differentiate the given equation with respect to a we find that $a^2=m/x$. Substitution of this value of a in the original equation gives $y^2=4mx$.
4. Let the legs lie along the positive x- and y-axes and let the x and y intercepts be $x = a$, $y = 0$, and $x = 0$, $y = b$. Then the hypotenuse lies along the line $y = (-b/a)x + b$. Since the area, $\frac{1}{2}$ ab, of the triangle is to be constant, we have $ab = k =$ constant. Using $a = k/b$ the family is given by $y = -(b^2x/k) + b$ where the parameter is b and k is constant. Then differentiation with respect to b gives $-(2bx/k) + 1 = 0$ and $b = k/2x$. If we substitute this value of b in the equation of the family we obtain $y = k/4x$.
5. The centers of the circles lie on the x-axis and hence may be denoted by $x = \alpha$, $y = 0$. The diameter of the circle is the length of the chord of the parabola at $x = \alpha$ and is thus $2\sqrt{\alpha}$. Thus the family of circles is $(x - \alpha)^2 + y^2 = \alpha$. Differentiation with respect to α yields $-2(x - \alpha) = 1$ or $\alpha = x + \frac{1}{4}$. If we substitute this value of α in the equation of the family we obtain $y^2 = x + \frac{1}{4}$.
6. Let us denote a point on the parabola $y = x^2$ by (α, β) to avoid confusion with the x and y of the envelope we seek. Then the center of any circle is (α, β) and its radius is β. Thus the equation of the family is $(x - \alpha)^2 + (y - \beta)^2 = \beta^2$ or $x^2 - 2\alpha x + \alpha^2 + y^2 - 2\beta y = 0$. Since $\beta = \alpha^2$, the family is $x^2 - 2\alpha x + \alpha^2 + y^2 - 2\alpha^2 y = 0$. If we differentiate with respect to α we obtain $-x + \alpha - 2\alpha y = 0$. If we substitute the value of α in the equation of the family and clear of fractions we find that we can group the terms thus: $(x^2 + y^2)(1 - 2y)^2 - 2x^2(1 - 2y) + x^2(1 - 2y) = 0$. If we divide through by $1 - 2y$ and simplify we get $y[x^2 + y^2 - (y/2)] = 0$. Hence the envelope consists of $y = 0$ and $x^2 + y^2 = y/2$. The factor $1 - 2y$ which was eliminated earlier is not part of the envelope but was introduced by the clearing of fractions.
7. The lines may be represented by $y = (-b/a)x + b$. (See Exercise 4.) Here $a + b = k$ and so the family is $y = [1 - (k/a)]x + k - a$. By differentiating with respect to a we find $a = \sqrt{kx}$ and $y = x - 2\sqrt{kx} + k$. Thus $y = (\sqrt{x} - \sqrt{k})^2$ from which the result follows with $\ell = \sqrt{k}$. The portion of the curve $y = (\sqrt{x} - \sqrt{k})^2$ which is the envelope comes from taking $\sqrt{y} = -(\sqrt{x} - \sqrt{k})$.

8. Any circle has the form $(x - x_0)^2 + (y - y_0)^2 = r^2$. Since the center (x_0, y_0) lies on the hyperbola $xy = 1$, we have $y_0 = 1/x_0$. Since $(0, 0)$ lies on the circle we leave $r^2 = x_0^2 + 1/x_0^2$. Thus letting $x_0 = \alpha$, the family of circles is $(x - \alpha)^2 + [y - (1/\alpha)]^2 = \alpha^2 + 1/\alpha^2$ or $x^2 - 2\alpha x + y^2 - 2y/\alpha = 0$. By differentiating with respect to α we obtain $\alpha = \sqrt{y/x}$ and by substituting this value in the equation of the family we obtain $(x^2 + y^2)^2 = 16xy$.
9. (a) Taking A as the parameter and by differentiating the equations $y - x \tan A + (16/V^2)(1 + \tan^2 A)x^2 = 0$ with respect to A we obtain $-x\sec^2 A + (16/V^2)(2\tan A \sec^2 A)x^2 = 0$. Since $\sec A$ is never zero we have $\tan A = V^2/32x$. By substituting the value obtained for $\tan A$ into the given family of trajectories, we obtain the parabola $y - (V^2/64) = (-16/V^2)x^2$.
   (b) To find the focus consider the parabola $y = -(16/V^2)x^2$. This is of the form $y = (-1/4p)x^2$ (see Chap. IV). Hence $p = V^2/64$. The focus of $y = (-1/4p)x^2$ is $(0, -p)$. Hence the focus of $y = -(16/V^2)x^2$ is $(0, -V^2/64)$. Now the y-values of the envelope are $V^2/64$ larger than those of the parabola just considered. Hence the focus of the envelope is $(0, 0)$.
10. In any vertical plane through the nozzle the section of the surface is a parabola of surety. Thus the full surface arises by rotating this parabola about its axis and the surface is a paraboloid of revolution. The equation of the surface is $z - (V^2/64) = (16/V^2)(x^2 + y^2)$ where the z-axis is the vertical axis.
11. (a) The slope of the tangent to the ellipse is $\dot{y}/\dot{x}$ where the dot means differentiation with respect to ϕ. Hence the slope is $-b\cos\phi/a\sin\phi$. Then the slope of the normal is $(a/b)\tan\phi$ and the equation of the family of normals is $y - b\sin\phi = (a/b)(\tan\phi)(x - a\cos\phi)$. We differentiate with respect to ϕ: $-b\cos\phi = (a/b)(\sec^2\phi)(x - a\cos\phi) + (a/b)\cdot\tan\phi\,(a\sin\phi)$. If we solve this equation for x and simplify we obtain $ax = (a^2 - b^2)\cos^3\phi$. If we substitute this value of x in the equation of the family we obtain $by = (b^2 - a^2)\sin^3\phi$. Thus we have the equations of the envelope in parametric form.
   (b) The method is the same as in (a). The equation of the family of normals is $y - b\tan\phi = -(a/b)(\sin\phi)(x - a\sec\phi)$. Differentiating with respect to ϕ and solving for x gives $ax = (a^2 + b^2)\sec^3\phi$. Substituting this value of x in the equation of the family yields $by = -(a^2 + b^2)\tan^3\phi$. Again we have the envelope in parametric form.
   (c) The method is the same as in (a). The equation of the family of normals is $y - a(1 - \cos\theta) = (\cot\theta - \csc\theta)[x - a(\theta - \sin\theta)]$. Differentiating with respect to θ and solving for x yields $x = a\theta + a\sin\theta$. Substituting this value of x in the equation of the family yields $y = -a + a\cos\theta$.

12. Differentiating the original equation with respect to b and solving for b gives $b = x+a/2$. Substituting this value of b in the equation of the family yields $y^2 = a(x+a/4)$ for the envelope.

13. The slope of the reflected rays is $\tan 2\theta$. Hence the equation of the family of reflected rays is $y - \sin\theta = \tan 2\theta(x - \cos\theta)$. Differentiating with respect to θ gives $-\cos\theta = 2\sec^2 2\theta(x - \cos\theta) + \tan 2\theta \sin\theta$ or $x = (-\cos\theta - 2\sec^2 2\theta \cdot \cos\theta - \tan 2\theta \sin\theta)/2\sec^2 2\theta$. Use of trigonometric identities gives $x = \cos\theta - (\cos 2\theta \cos\theta)/2$. This expression can be converted to $x = [(3\cos\theta)/4)] - (1/4)\cos 3\theta$ and substitution of this value of x in the equation of the family gives the text's expression for y. Of course there are other valid expressions for x and y as the parametric equations of the envelope.

Solutions to Chapter 23

CHAPTER 23, SECTION 2

1. In each case we denote the given double integral by G and the inner integral by I.

   (a) $I = \int_0^1 (x + y)\,dy = xy + (y^2/2)\Big|_{y=0}^{y=1}$. Then $G = \int_0^2 (x + \tfrac{1}{2})\,dx$
   $= x^2/2 + (x/2)\Big|_0^2 = 3.$

   (b) We have $I = \int_0^y y\,dx = yx\Big|_{x=0}^{x=y} = y^2$. Thus $G = \int_{1/2}^1 y^2\,dy = \tfrac{7}{24}$.

   (c) We have $I = \int_{x^3-x}^{3x-x^3} 1\,dy = y\Big|_{x^3-x}^{3x-x^3} = 4x - 2x^3$. Thus $G = \int_0^{\sqrt{2}} (4x - 2x^3)\,dx = 2.$

   (d) We have $I = \int_t^{10t} \sqrt{st - t^2}\,ds$. Since t is treated as a constant in this integral, this integral may be put in the form of #1 in the integral tables with $u = ts - t^2$, $du = t\,ds$, $n = \tfrac{1}{2}$. Thus $I = (1/t)\int_t^{10t} \sqrt{ts - t^2}\,t\,ds$
   $= (1/t)\,\tfrac{2}{3}(ts - t^2)^{3/2}\Big|_{s=t}^{s=10t} = 18t^2$. Finally $G = \int_0^b 18t^2\,dt = 6b^3$.

   (e) $I = \int_{\sqrt{y}}^{2-y} y^2\,dx = y^2 x\Big|_{x=\sqrt{y}}^{x=2-y} = 2y^2 - y^3 - y^{5/2}$. Then $G = \int_0^1 (2y^2 - y^3 - y^{5/2})\,dy$
   $= \tfrac{2}{3}y^3 - y^4/4 - \tfrac{2}{7}y^{7/2}\Big|_0^1 = \tfrac{11}{84}$.

2. We observe that since, by (1), z is symmetric in x and y, the integrals (13) and (14) are identical if we interchange x and y. Thus the evaluation of (14) is obtained in exactly the same way as that of (13) with the roles of x and y interchanged.

3. The required volume lies entirely above the z-plane and is the same in all four quadrants. Since the given surface cuts the z-plane along the ellipse $(x^2/a^2) + (y^2/b^2) = 1$, we find $V = 4\int_0^a \left[\int_0^{b\sqrt{1-x^2/a^2}} (1 - x^2/a^2 - y^2/b^2)\,dy\right]dx$.
   Evaluation of the inner integral gives $(y - x^2y/a^2 - y^3/3b^2)\Big|_{y=0}^{y=b\sqrt{1-x^2/a^2}}$
   $= 2b(a^2 - x^2)^{3/2}/3a^3$. Thus $V = (8b/3a^3)\int_0^a (a^2 - x^2)^{3/2}\,dx$. Use of the integral formula #31 leads to $V = \pi ab/2$.

4. From $y = 0$ to $y = 9$, x varies between 0 and the parabola $y = x^2$. From $y = 9$ to $y = 18$, x varies between 0 and the parabola $y = 18 - x^2$. Thus following the suggestion, we have $V = \int_0^9 \int_0^{\sqrt{y}} z\,dx\,dy + \int_9^{18} \int_0^{\sqrt{18-y}} z\,dx\,dy$. Since $z = xy$, we obtain $V = \int_0^9 \frac{1}{2} y^2\,dy + \int_9^{18} \frac{1}{2} y(18 - y)\,dy = \frac{729}{2}$.

5. We have $\int_A\int (2xy - x^2)\,dx\,dy = \int_0^4 \int_{-1}^2 (2xy - x^2)\,dx\,dy$. Successive integrations gives the answer of 12.

6. We have (see Fig. #1), $\int_A\int (y - 2x)\,dx\,dy$ $= \int_0^6 \int_{(y-6)/3}^0 (y - 2x)\,dx\,dy = 20$.

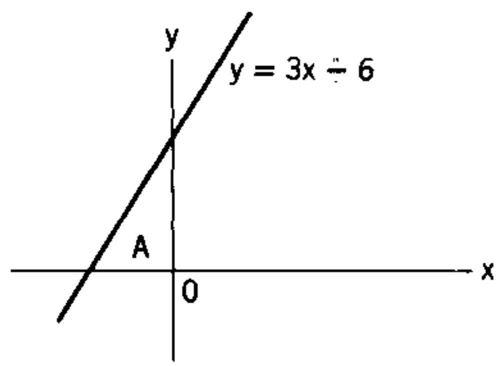


Fig. 1

7. Since the plane, $z = 9 - x - y$, lies above all points of the triangle, we have (see Fig. #2) $V = \int_A\int (9 - x - y)\,dy\,dx$ $= \int_0^3 \int_0^{2x/3} (9 - x - y)\,dy\,dx = 19$.

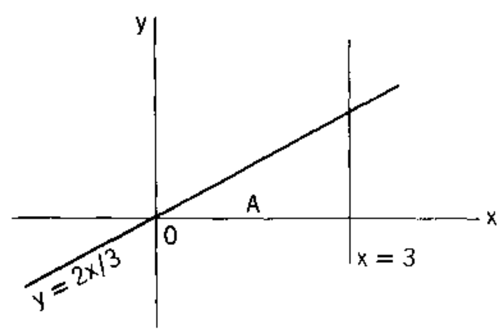


Fig. 2

8. Since $z \geq 0$ for $y \geq 0$, the plane lies above all points of A (see Fig. #3). Thus $V = \int_A \int 2y\,dy\,dx = \int_3^6 \int_0^{\sqrt{36-x^2}} 2y\,dy\,dx = 45$.

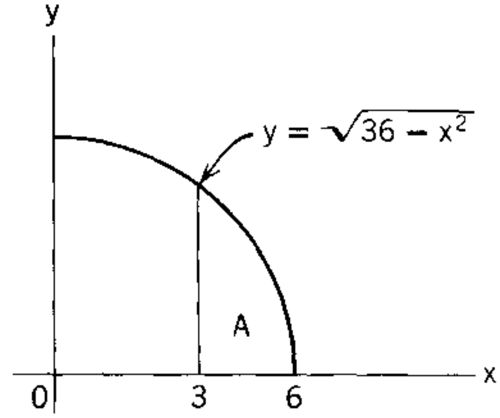


Fig. 3

9. We have (see Fig. #4), $V = \int_A \int z\,dx\,dy$ $= \int_0^1 \int_{-3\sqrt{1-y}}^{3\sqrt{1-y}} \sqrt{y}\,dx\,dy = 6\int_0^1 \sqrt{y - y^2}\,dy$. Use of integral formula #56 leads to $V = 3\pi/4$.

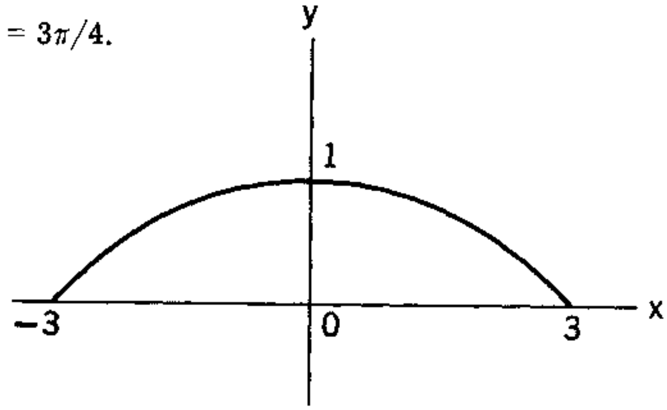


Fig. 4

10. Since $z \geq 0$ for (x, y) in the first quadrant, we have (see Fig. #5) $V = \int_A \int xy\,dx\,dy$ $= \int_0^{1/\sqrt{2}} \int_y^{\sqrt{1-y^2}} xy\,dx\,dy = 1/16$.

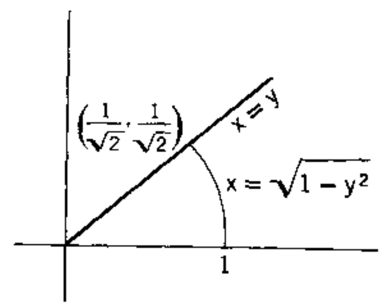


Fig. 5

11. By the usual reasoning $V = \int_0^2 \int_0^{3\sqrt{4-y^2}/2} (x + y)\,dx\,dy = 10$.

12. Following the suggestion, we have $V = \int_A \int y\,dx\,dy$. Using Figure #6 and the symmetry of the paraboloid, the paraboloid cuts the xz-plane where $y = 0$, we have $V = 4\int_0^4 \int_0^{\sqrt{16-x^2}/2} (2 - x^2/8 - z^2/2)\,dz\,dx$ $= \frac{1}{6}\int_0^4 (16 - x^2)^{3/2}\,dx$. Use of integral formula #31 leads to $V = 8\pi$.

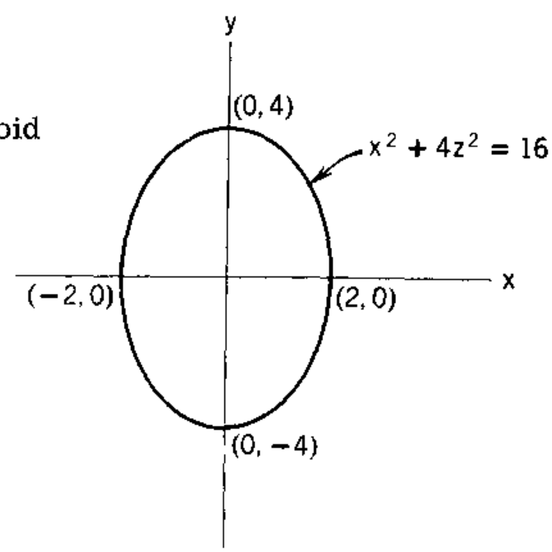


Fig. 6

13. The plane cuts the xy-plane along $2x + 3y = 6$. In the usual way we obtain $V = \int_0^2 \int_0^{3-(3/2)y} (1 - x/3 - y/2)\,dx\,dy = 1$.

14. We note that $z \geq 0$ only for $|x| \leq 2$. Thus the volume in the first octant (i.e., $x \geq 0$, $y \geq 0$, $z \geq 0$) is given by (see Fig. #7) $V = \int_A \int z\,dy\,dx$ $= \int_0^2 \int_0^{(12-4x)/3} (4 - x^2)\,dy\,dx = 16$.

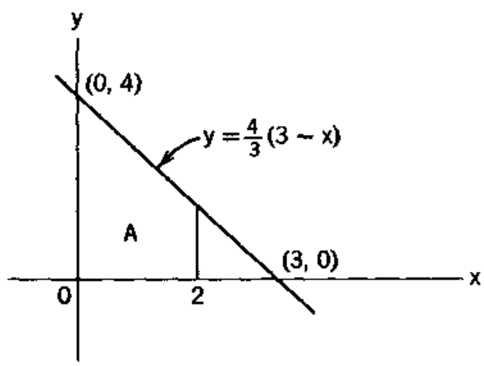


Fig. 7

15. (a) The volume in the first octant bounded by the cylinder $x^2 + z^2 = a^2$, the coordinate planes $y = 0$ and $z = 0$, and the plane $y = x$. See Fig. #8.

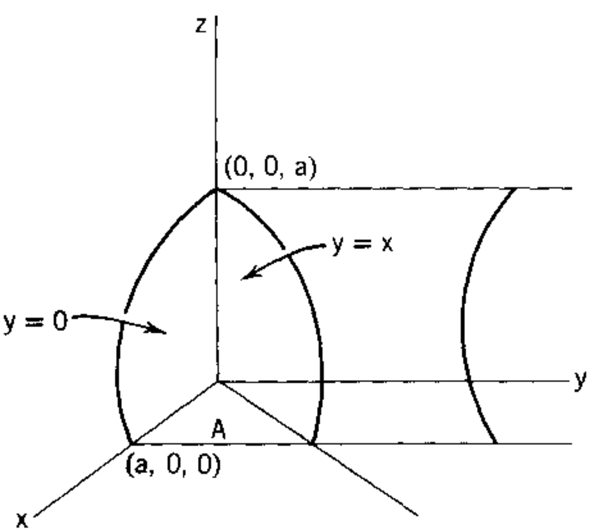


Fig. 8

(b) The volume under the plane $x + y + z = 2a$ and above the quarter circle $x^2 + y^2 = a^2$ in the first quadrant of the xy-plane.

16. In each case the integral is over a given area. Once a geometrical description of the area is obtained, it is possible to read off the limits of integration for integration in the reverse order.

(a) The area is that enclosed between the parabola $y = x - x^2/a$ and $y = x^2/a$ for $0 \leq x \leq a/2$. Thus we obtain Fig. 9. In particular the parabolas intersect at (0, 0) and (a/2, a/4). To integrate with respect to x, we need the value of x in terms of y on the left side of the parabola $y = x - (x^2/a)$ and on the right side of the parabola $y = x^2/a$. These values are, respectively, $x = a/2 - \sqrt{a^2/4 - ay}$ and $x = +\sqrt{ay}$. Noting that y varies from $y = 0$ to $y = a/4$, we have the answer in the text.

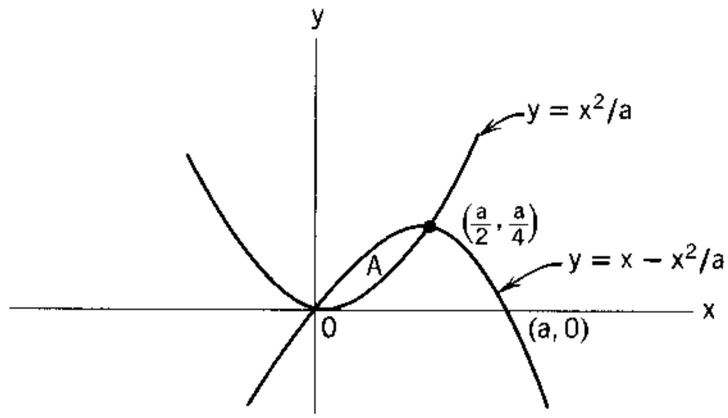


Fig. 9

(b) The area is that enclosed between the top side of the ellipse $x^2/a^2 + y^2/b^2 = 1$ and the line $y = b$ for $c \leq x \leq a$. Thus we obtain Fig. 10. To integrate with respect to x, we need the value of x on the right side of the ellipse. This value is $x = a\sqrt{b^2 - y^2}/b$. The point P in Fig. 10 lies on the ellipse and has x-coordinate c. Thus we find that P is $(c, (b/a)\sqrt{a^2 - c^2})$. We notice that the left hand x value of integration is different according as y is above or below the point P. For y above

P, we have $c \le x \le a$, $b\sqrt{a^2 - c^2}/a \le y \le b$. For y below P, we have $a\sqrt{b^2 - y^2}/b \le x \le a$, $a \le y \le b\sqrt{a^2 - c^2}/a$. Thus the answer is

$\int_{b\sqrt{a^2-c^2}/a}^{b} \int_c^a f(x, y)\,dx\,dy + \int_a^{b\sqrt{a^2-c^2}/a} \int_{a\sqrt{b^2-y^2}/b}^{a} f(x, y)\,dx\,dy.$

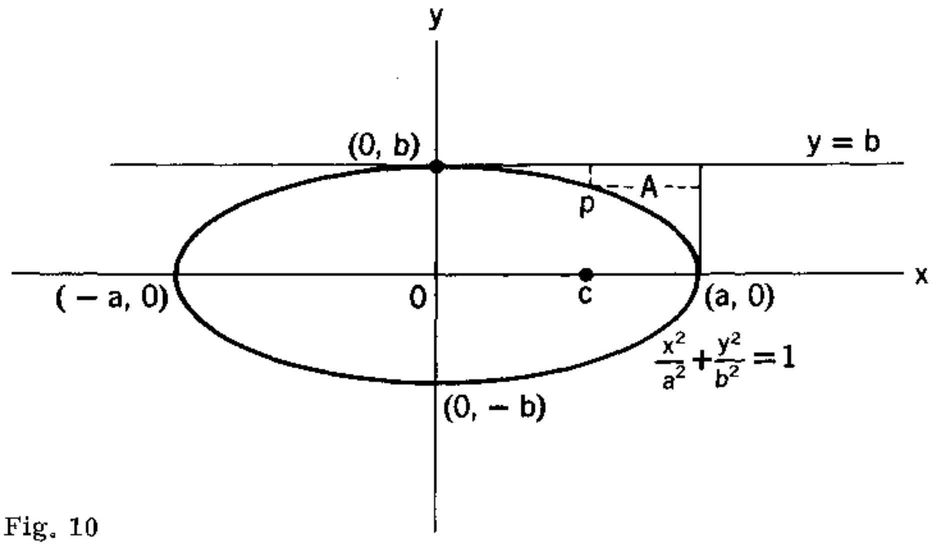


Fig. 10

(c) The area is shown in Fig. 11. The left-hand value of x is $\sqrt{a^2 - 4y^2}$ and the right-hand value is $\sqrt{a^2 - y^2}$ for $0 \le y \le a/2$. But for $a/2 \le y \le a$, the left-hand value is 0 and the right-hand value is as before. Hence

$\int_0^{a/2} \int_{\sqrt{a^2-4y^2}}^{\sqrt{a^2-y^2}} f(x, y)\,dx\,dy + \int_{a/2}^{a} \int_0^{\sqrt{a^2-y^2}} f(x, y)\,dx\,dy.$

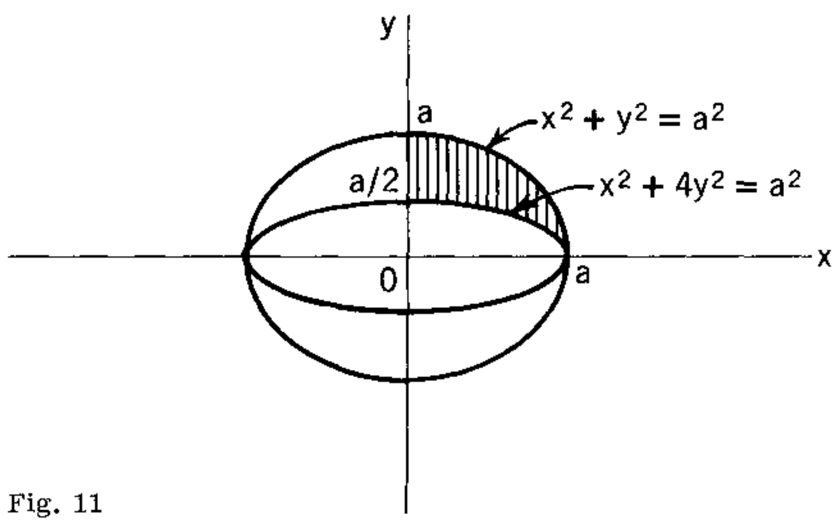


Fig. 11

17. (a) Imagine y increased a small amount so that V increases by ΔV. One can see geometrically that $\Delta V/\Delta y$ is an average increase in V which is a "slice" like PQRS but to the right. As Δy approaches 0, the slice approaches PQRS.
(b) Since $\partial V/\partial y$ is area PQRS, an increase in x only increases the area of PQRS and we can go back to our old result that $\dot{A} = y$ to conclude that $\partial(\partial V/\partial y)/\partial x$ is the "y-value" of the area or in this case is PQ.
(c) The argument is the same as (a) but with x and y reversed.
(d) The argument is the same as (b) but applied to PQMN. The answer is PQ.

18. The desired volume lies under the plane $z = mx$ and over the half of the circle $x^2 + y^2 = r^2$ for which x is positive because the plane goes through the v-axis and the wedge lies on the positive x-side. Hence y runs from $-\sqrt{r^2 - x^2}$ to $\sqrt{r^2 - x^2}$ and x runs from 0 to r. Then $V = \int_0^r \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} mx\,dy\,dx = 2mr^3/3$.

19. One can set up the integral for the volume as $\int_0^b \int_0^{\sqrt{b^2-x^2}} \sqrt{a^2 - x^2 - y^2}\,dy\,dx$ and attempt to use Formula 26 of the Integral Tables. This leads to the arc sin term, $\sqrt{b^2 - x^2}/\sqrt{a^2 - x^2}$. This cannot be integrated readily (if at all). However we avoid this difficulty if we first recognize that the desired volume consists of the volume of a cylinder which (above the xy-plane) runs from 0 to $z = \sqrt{a^2 - b^2}$ and then a spherical cap. The volume of the cylinder is $\pi b^2\sqrt{a^2 - b^2}$ and the total for the two portions (above and below the xy-plane) is $2\pi b^2\sqrt{a^2 - b^2}$.

To find the volume of the spherical cap regard it as lying above the yz-plane so that the surface is given by $x = \sqrt{a^2 - y^2 - z^2}$. The domain of integration in the yz-plane is given by y running from 0 to $\sqrt{a^2 - z^2}$ and z running from $\sqrt{a^2 - b^2}$ to a. Then the volume of this portion of the cap (which is $\frac{1}{8}$ the volume of the entire cap above and below the xy-plane) is given by $\int_{\sqrt{a^2-b^2}}^{a} \int_0^{\sqrt{a^2-z^2}} \sqrt{a^2 - y^2 - z^2}\,dy\,dz$. Now if we use Formula 26 of the Tables the integration is straightforward. The result is $(\pi/4)[(2a^3/3) - a^2\sqrt{a^2 - b^2} + (a^2 - b^2)^{3/2}/3]$. Multiply this by 8 and add the volume of the cylinder.

20. The desired volume lies under the plane $z = x$ and over the portion of the circle $x^2 + y^2 = 9$ in the (xy)-plane which is cut off by the planes $y = 0$ and $z = x$. Since the trace of $z = x$ in the (x, y)-plane is $x = 0$, this is just the first quadrant of the circle, thus $V = \int_0^3 \int_0^{\sqrt{9-y^2}} x\,dx\,dy = 9$.

21. The desired volume lies under the plane $x + z = 1$ and over the portion of the parabola $y^2 = x$ in the (xy)-plane which is cut off by the planes $x + z = 1$ and $y = 0$. Thus (see Fig. #12) $V = \int_0^1 \int_{y^2}^1 (1 - x)\,dx\,dy = 4/_{15}$.

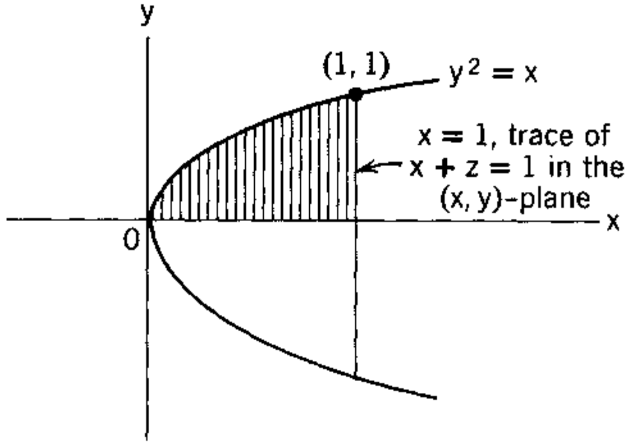


Fig. 12

22. By reasoning as in #21 (see Fig. #13) we obtain
$V = \int_0^1 \int_{y^2}^{2-y} (2 - x - y)\,dx\,dy = 17/_{20}$.

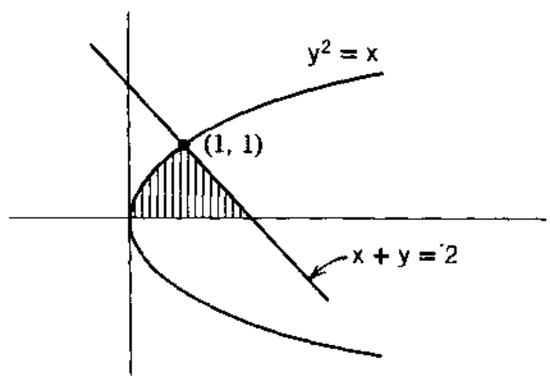


Fig. 13

23. Since the volume (Fig. 14) is symmetric with respect to the (x, y)-plane, it is twice the volume obtained for $z \geq 0$. This latter volume lies below $z = \sqrt{r^2 - x^2}$ and above the portion of the circle $x^2 + y^2 = r^2$ in the (x, y)-plane cut off by $x^2 + z^2 = r^2$. Since the trace of $x^2 + z^2 = r^2$ in the x, y-plane is $x = \pm r$, we actually must integrate over the entire circle. Thus
$V = 2 \int_{-r}^{r} \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \sqrt{r^2 - x^2}\,dy\,dx = 8 \int_0^r \int_0^{\sqrt{r^2-x^2}} \sqrt{r^2 - x^2}\,dy\,dx = 16r^3/3$.

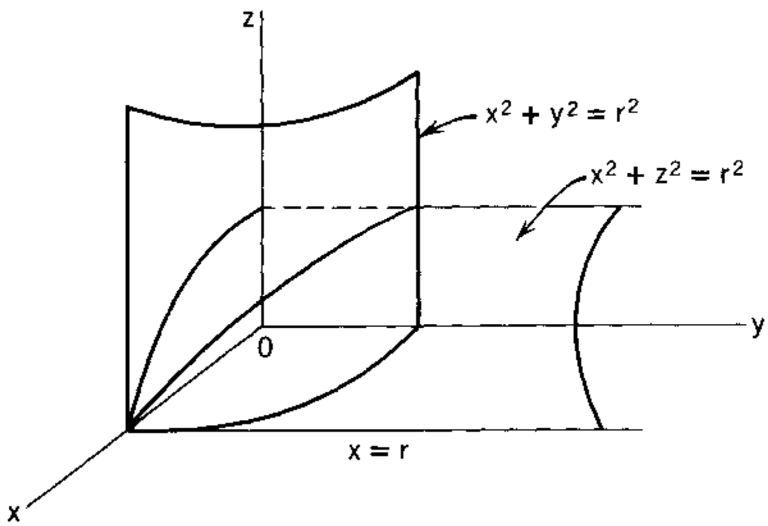


Fig. 14

24. See Fig. 15. $V = \int_0^1 \int_{x^2}^{\sqrt{x}} (12 + y - x^2)\,dy\,dx = {}^{569}/_{140}$.

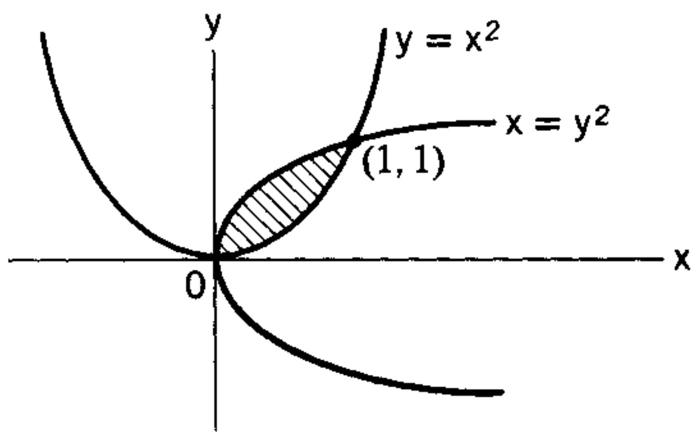


Fig. 15

25. The area integrated over is shown in Fig. 16. Thus the answer is $\int_0^a \int_y^a f(x, y)\,dx\,dy$.

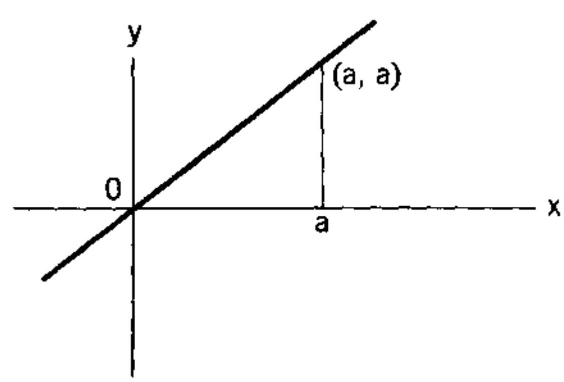


Fig. 16

26. Let the usual u of the integration by parts formula be denoted by w so that no confusion of variables results. Then the formula reads $\int w\,dv = wv - \int v\,dw$. Since we are dealing with a definite integral the formula reads $\int_0^x w\,dv = wv\Big|_0^x - \int_0^x v\,dw$. Now to integrate $\int_0^x f(u)(x-u)\,du$ let $w = x - u$, $dv = f(u)\,du$. Then $dw = -1$ and $v = \int_0^u f(u)\,du$ which is better written as $\int_0^u f(t)\,dt$ to avoid confusion between the variable of integration and the upper end value. Then $\int_0^x f(u)(x-u)\,du = (x-u)\int_0^u f(t)\,dt\Big|_0^x + \int_0^x\int_0^u f(t)\,dt\,du$. The first term on the right side vanishes because when we substitute x for u, the factor $x - u$ is 0 and when we substitute 0 for u the other factor is 0.

CHAPTER 23, SECTION 3

1. In each case a sketch of the curves and calculation of their common points leads at once to the stated limits of integration.
   (a) $A = \int_0^1 \int_y^{y^{2/3}} dx\,dy = 1/10$.
   (b) $A = \int_{-3}^3 \int_{y^2/9}^{\sqrt{10-y^2}} dx\,dy = 1 + 10\sin^{-1}(3/\sqrt{10})$. Suggestion: use integral formula #26.
   (c) $A = \int_{-2}^1 \int_{y^2-1}^{1-y} dx\,dy = 4\tfrac{1}{2}$.
   (d) $A = \int_{-2}^1 \int_{x+7}^{9-x^2} dy\,dx = 4\tfrac{1}{2}$.
   (e) $A = \int_1^4 \int_{4/y}^{5-y} dx\,dy = 7\tfrac{1}{2} - 4\log 4$.
   (f) $A = \int_{-2}^2 \int_{y^2/4}^{5-y^2} dx\,dy = 13\tfrac{1}{3}$.
   (g) $A = \int_0^2 \int_{3x^2-6x}^{2x-x^2} dy\,dx = 5\tfrac{1}{3}$.
   (h) $A = 2\int_0^2 \int_{x^3-3x}^{x^{3/4}} dy\,dx = 6$.
2. By (19), $M/4 = (k/3)\int_0^a (a^2 - x^2)^{3/2}\,dx$. Using integral formula #31, we find $M/4 = k\pi a^4/16$.
3. Choose the fixed diameter as lying along the x-axis and use $D = ky$ in (18) to obtain $M = \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} ky\,dy\,dx = 4k\int_0^a\int_0^{\sqrt{a^2-x^2}} y\,dy\,dx = 4ka^3/3$.

4. Choose the legs (say of length a) along the x- and y-axes. Then $D = k(x^2 + y^2)$ and, by (18), $M = \int_A \int k(x^2 + y^2)\,dx\,dy$. Since the equation of the hypotenuse is $x + y = a$, we obtain $M = k \int_0^a \int_0^{a-y} (x^2 + y^2)\,dx\,dy = ka^4/6$.
5. Choose two sides along the x- and y-axes. Then, as in the previous example, $D = k(x^2 + y^2)$. Then $M = k \int_0^a \int_0^a (x^2 + y^2)\,dx\,dy = 2ka^4/3$.
6. Choose the legs along the x- and y-axes. Then $D = ky$ and $M = k \int_0^a \int_0^{a-y} y\,dx\,dy = ka^3/6$.
7. Here $D = ky$. Then $M = k \int_0^1 \int_0^{e^x} y\,dy\,dx = k[e^2 - 1]/4$.
8. Here $D = k(4 - y)$. Then $M = k \int_{-8}^{8} \int_{x^2/16}^{4} (4 - y)\,dy\,dx = 1024k/15$.
9. Here $D = k(x + y)$. Then $M = k \int_0^5 \int_0^{\sqrt{25-y^2}} (x + y)\,dx\,dy = 250k/3$.

CHAPTER 23, SECTION 5, FIRST SET

In each case the equations $\rho = \sqrt{x^2 + y^2}$, $\sin\theta = y/\sqrt{x^2 + y^2}$, $\cos\theta = x/\sqrt{x^2 + y^2}$, may be used to express the given figure in rectangular coordinates if this transformation is needed to identify the figure.

(a) Cylinder; axis is the z-axis; radius is 1.
(b) Plane perpendicular to xy-plane ($z = 0$ or $\rho\theta$-plane) bisecting 1st and 3rd quadrants of the xy-plane.
(c) Sphere; center is the origin; radius is 2.
(d) Cone with two nappes; vertex at origin; axis is z-axis.
(e) Plane parallel to and 4 units in front of the yz-plane.
(f) Plane parallel to and 3 units in front of xz-plane.
(g) Cylinder; axis is parallel to the z-axis, radius is 2; center is at $\rho = 2$, $\theta = 0$.
(h) Cylinder; axis is parallel to the z-axis, radius is 1; center is at $\rho = 1$, $\theta = 90°$.
(i) Plane parallel to xy-plane and 5 units above it.

CHAPTER 23, SECTION 5, SECOND SET

1. Denote the given integral by G and the inner integral by I.

(a) We have $I = \int_0^{a\cos\theta} \sin\theta\,\rho\,d\rho = \sin\theta(\rho^2/2)\Big|_{\rho=0}^{\rho=a\cos\theta} = (a^2\cos^2\theta\sin\theta)/2$.

Thus $G = (a^2/2)\int_0^{\pi} \cos^2\theta\sin\theta\,d\theta = a^2/3$.

(b) We have $I = \int_0^{a(1+\cos\theta)} \rho^2 \sin\theta\, d\rho = \sin\theta\, a^3(1+\cos\theta)^3/3$. Thus

$G = (a^3/3)\int_0^{\pi} (1+\cos\theta)^3 \sin\theta\, d\theta = 4a^3/3$.

(c) We have $I = \int_{a\cos\theta}^{a} \rho^4\, d\rho = a^5(1-\cos^5\theta)/5$. Thus

$G = (a^5/5)\int_0^{\pi/2} (1-\cos^5\theta)\, d\theta = (a^5/5)[(\pi/2) - \int_0^{\pi/2}\cos^5\theta\, d\theta]$.

Use integral formula #73 twice to obtain $G = (a^5/10)(\pi - {}^{16}/_{15})$.

2. The formula for area is $A = \int_A\int \rho\, d\rho\, d\theta$. See Fig. 17 and note that the curves intersect for $2a\cos\theta = a$, that is, for $\theta = \pm 60°$. Then $A = 2\int_0^{\pi/3}\int_a^{2a\cos\theta} \rho\, d\rho\, d\theta = a^2[(\pi/3)+(\sqrt{3}/2)]$.

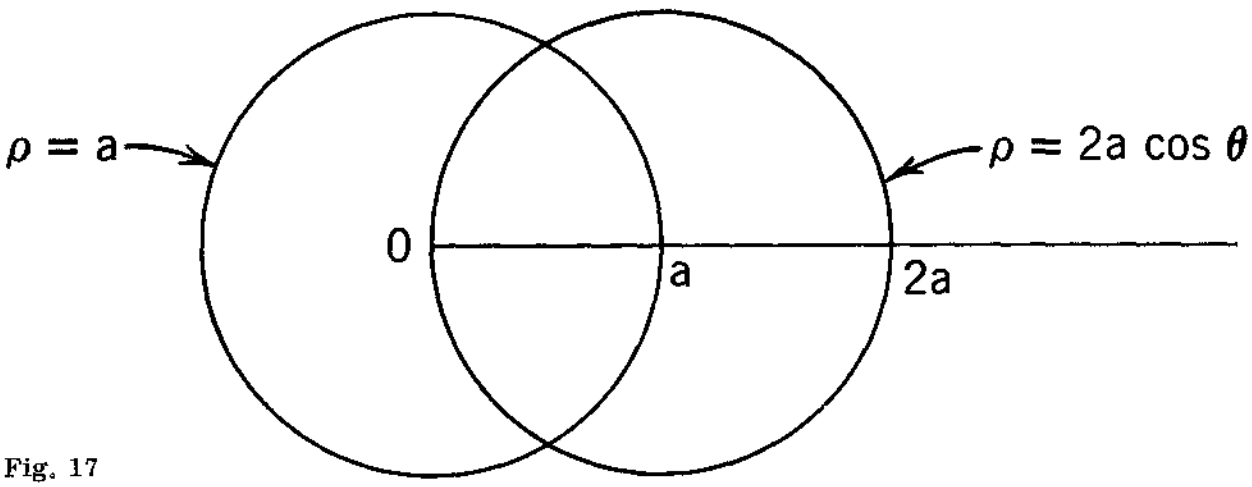


Fig. 17

3. See Figure 18. $A = 2\int_0^{\theta_0}\int_{3/4\cos\theta}^{1} \rho\, d\rho\, d\theta$, where $\theta_0 = \arccos {}^3/_4$. Then

$A = 2\int_0^{\theta_0} ({}^1/_2 - {}^9/_{32}\sec^2\theta)\, d\theta = (\theta - {}^9/_{16}\tan\theta)_0^{\theta_0} = \arccos {}^3/_4 - {}^9/_{16}(\sqrt{7}/3)$.

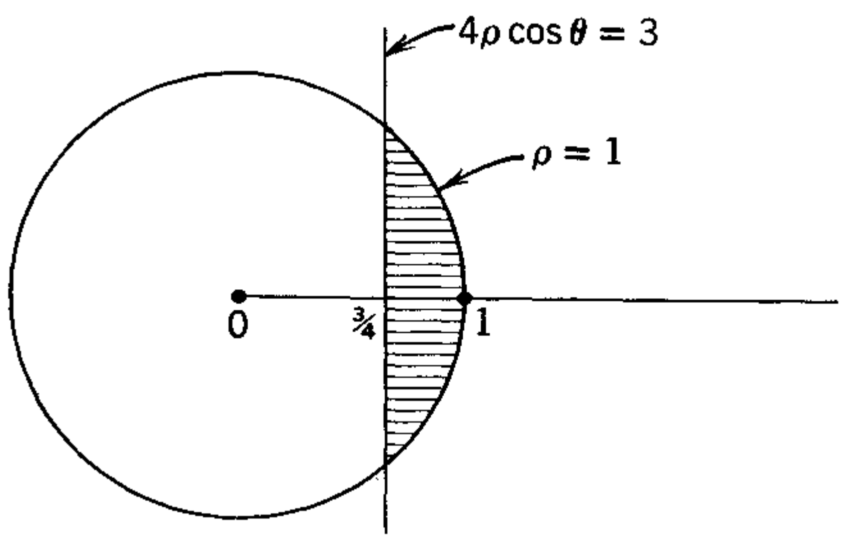


Fig. 18

4. See Figure 19. We have $A = \int_{-\pi}^{\pi} \int_{\cos\theta}^{3\cos\theta} \rho \, d\rho \, d\theta = 2\pi$. This area may also be computed by elementary means.

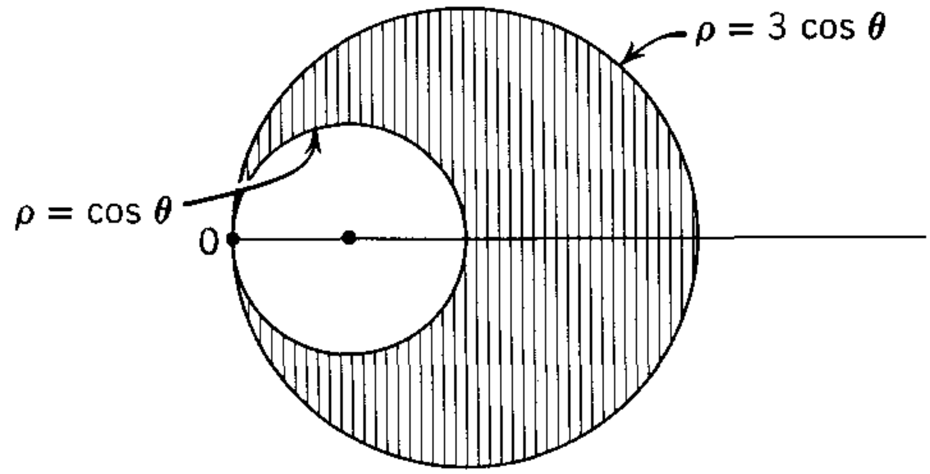


Fig. 19

5. See Figure 20. Since the curves intersect for $\cos\theta = \frac{1}{2}$, we have $A = 2 \int_0^{\pi/3} \int_{3/4\cos\theta}^{1+\cos\theta} \rho \, d\rho \, d\theta = (\pi/2) - (9\sqrt{3}/16)$.

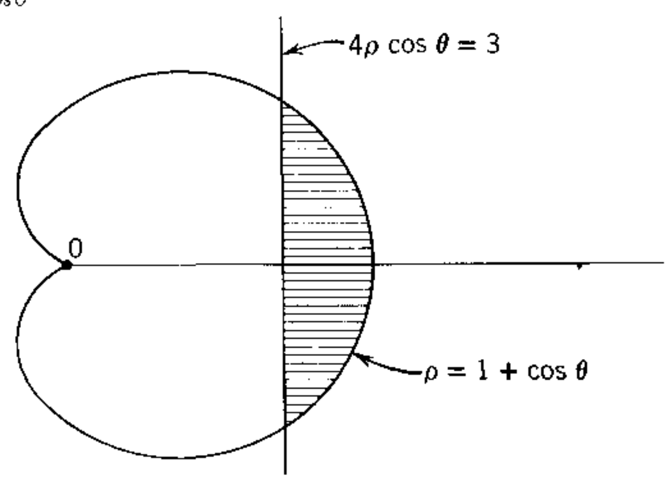


Fig. 20

6. See Figure 21 and note that the curves intersect at $\theta = \pm 90°$. Then $A = 2 \int_0^{\pi/2} \int_{1/(1+\cos\theta)}^{1} \rho \, d\rho \, d\theta = \int_0^{\pi/2} [1 - 1/(1 + \cos\theta)^2] \, d\theta$. To integrate $1/(1 + \cos\theta)^2$ there are several methods. One is to use the fact that the expression is a rational function of $\cos\theta$ (see Chapter 13, Section 5). Or one may use the identity $\cos(\theta/2) = \sqrt{(1 + \cos\theta)/2}$ to obtain $(1 + \cos\theta)^2 = 4\cos^4(\theta/2)$. Then $1/(1 + \cos\theta)^2 = \frac{1}{4}\sec^4(\theta/2) = \frac{1}{4}\sec^2(\theta/2)[1 + \tan^2(\theta/2)]$. This expression is immediately integrable, the first, $\sec^2\theta/2$, because it is the derivative of $\tan\theta/2$ (apart from constant factors) and the second because if we let $u = \tan(\theta/2)$ then $du/d\theta = \sec^2(\theta/2)$. The final result is $\pi/2 - \frac{2}{3}$.

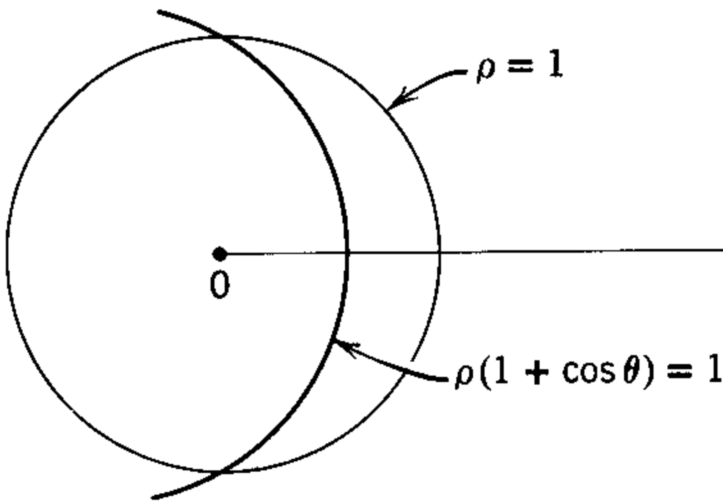


Fig. 21

7. See Figure 22 and note that the curves intersect for $\theta = \pm 30°, \pm 150°$. Thus $A = 4\int_0^{\pi/6}\int_a^{\sqrt{2a^2\cos 2\theta}} \rho\, d\rho\, d\theta = a^2(\sqrt{3} - \pi/3) = 0.684a^2$.

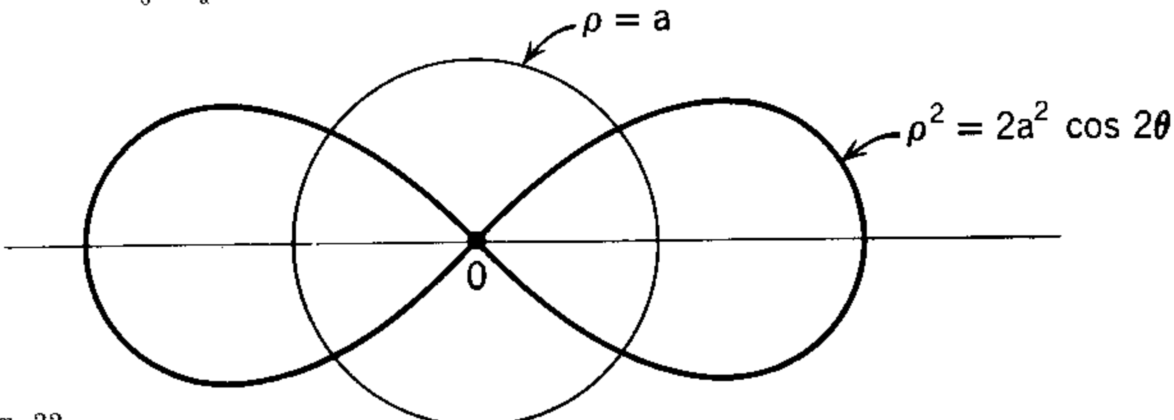


Fig. 22

8. If D is the density then $M = \int\int_A D(\rho,\theta)\, dA = \int\int_A D(\rho,\theta)\rho\, d\rho\, d\theta$. If we place the circle so that it goes through the pole and in fact has the equation $\rho = 2a\cos\theta$, then we may take the pole to be the fixed point on the circumference and then $D = k\rho^2$. Then the mass is given by twice the mass of the upper half, which leads at once to the text's answer.
9. If a surface is given by $z = f(\rho,\theta)$ and lies above an area A in the xy-plane, then the volume under the surface and above A is $V = \int\int_A f(\rho,\theta)\, dA$ $= \int\int_A f(\rho,\theta)\rho\, d\rho\, d\theta$. Thus in the present case, we find $V = \int_0^{\pi/2}\int_0^a \sqrt{a^2 - \rho^2}\,\rho\, d\rho\, d\theta = \pi a^3/6$.

10. See Exercise 8. Here we take the boundary circles to be $\rho = a$ and $\rho = b$, $b > a$. We have $M = \int_0^{2\pi}\int_a^b (k/\rho)\rho \, d\rho \, d\theta = 2\pi k(b - a)$.
11. See Exercise 9. Here the surface lies above the circle $\rho = 2$ in the $\rho\theta$-plane. We have $V = \int_0^{\pi/2}\int_0^2 \frac{1}{2}\sqrt{36 - 9\rho^2}\rho \, d\rho \, d\theta = 2\pi$.
12. See Exercises 2 and 8. We have $M = \int_{-\pi/3}^{\pi/3}\int_a^{2a\cos\theta} \sin\theta \, \rho \, d\rho \, d\theta = 2a^2/3$.
13. As pointed out in Exercise 9, volume in cylindrical coordinates is $\iint_A z \, dA = \iint_A f(\rho, \theta)\rho \, d\rho \, d\theta$. In the present case the surface is $x^2 + y^2 + z^2 = 4a^2$ or $\rho^2 + z^2 = 4a^2$ so that $z = \sqrt{4a^2 - \rho^2}$. The next question is, what does the cylinder cut out of the volume of the sphere? The cylinder is parallel to the z-axis and cuts the xy-plane or $\rho\theta$-plane in a circle $(x - a^2) + y^2 = a^2$. The same circle is described in ρ and θ coordinates as $\rho = 2a\cos\theta$. This circle lies inside the circle $x^2 + y^2 = 4a^2$ in which the sphere cuts the xy-plane or $\rho\theta$-plane and in fact the two circles are tangent at $\rho = 2a$ and $\theta = 0$ (Fig. 23). Hence the volume we want lies above the circle $\rho = 2a\cos\theta$. Then $V = 2\int_0^{\pi/2}\int_0^{2a\cos\theta} \sqrt{4a^2 - \rho^2}\rho \, d\rho \, d\theta$. The integrations are straightforward and the answer is $(16a^3/3)(\pi/2 - 2/3)$.

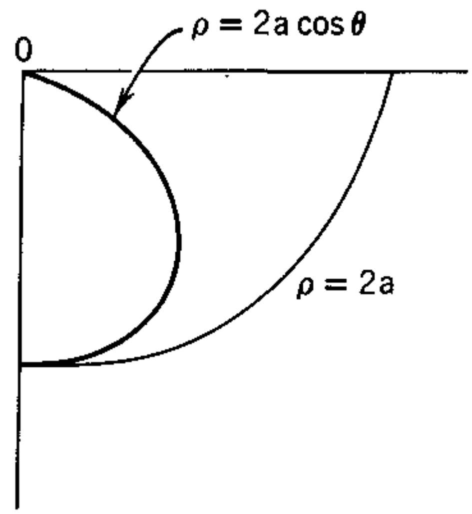


Fig. 23

14. A homogeneous lamina is one with a constant density so that $D = k$. The mass M is given by $\iint_A D\rho \, d\rho \, d\theta$. The cardioid is described as θ varies from 0 to 2π. Hence $M = \int_0^{2\pi}\int_0^{a(1+\sin\theta)} k\rho \, d\rho \, d\theta$. The integration is straightforward and $M = 3\pi ka^2/2$.
15. Using the suggestion and the relations $\cos\psi = \ell/r$, $r^2 = \ell^2 + \rho^2$, we have $F = \int_0^{2\pi}\int_0^a [(GMt\ell\rho)/(\ell^2 + \rho^2)^{3/2}] d\rho \, d\theta = 2\pi GMt\ell \int_0^a [\rho \, d\rho/(\ell^2 + \rho^2)^{3/2}]$ $= 2\pi GMt(1 - \ell/\sqrt{a^2 + \ell^2})$.
16. The idea of regarding a solid as a sum of discs and, knowing the properties of the discs, deriving a fact about the solid is good and we did do such Exercises in Chap. 16, Section 6. However here the property of the disc depends on two variables y and z. To write a double integral, y and z must vary independently in some two-dimensional domain; that is, within the two-dimensional domain any value of y must occur with any value of z. In

the present case the allowable y and z are related by $y^2 = 4z$. Hence y and z are not independent or, one can say, the domain of allowable y and z values is a one-dimensional curve.

17. The integral can be considered as the volume under the surface $z = e^{-x^2-y^2}$ and above the xy-plane. Thus $V = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-x^2-y^2}\,dx\,dy = \iint_A e^{-x^2-y^2}\,dA$. Replacing dA by its value in cylindrical coordinates, we obtain
$V = \int_0^{2\pi}\int_0^{\infty} e^{-\rho^2}\rho\,d\rho\,d\theta = 2\pi\int_0^{\infty} e^{-\rho^2}\rho\,d\rho = -\pi e^{-\rho^2}\Big|_0^{\infty} = \pi.$

18. The double integral is $E = k\int_0^{2\pi}\int_0^{\rho}\rho d\rho d\theta \,/\, \rho^m = k\int_0^{2\pi}\int_0^{\rho}\rho^{1-m}d\rho d\theta =$
$2\pi k\rho^{2-m}/(2-m)$.

CHAPTER 23, SECTION 6

1. In each case we denote the innermost integral by I, the second inner integral by H and the given integral by G.

(a) $I = \int_2^4 x^2y^2z\,dz = x^2y^2\int_2^4 z\,dz = 6x^2y^2$. $H = \int_0^1 6x^2y^2\,dy = 6x^2\int_0^1 y^2\,dy = 2x^2$.
$G = 2\int_1^2 x^2\,dx = {}^{14}\!/_3$.

(b) $I = \int_0^{2-x} dz = 2 - x$. $H = \int_0^{\sqrt{2x-x^2}} (2-x)\,dy = (2-x)\sqrt{2x-x^2}$.
$G = \int_0^1 (2-x)\sqrt{2x-x^2}\,dx$. We should like to let $u = 2x - x^2$.
Then $du/dx = 2 - 2x$. If we multiply G by 2 and divide by $1/2$, we have
$G = 1/2\int_0^1 \sqrt{2x-x^2}\,(4-2x)\,dx = 1/2\int_0^1\sqrt{2x-x^2}\,(2-2x)\,dx + \int_0^1\sqrt{2x-x^2}\,dx$.
The first integrand is of the form $u^{1/2}du/dx$. The second can be integrated by using formula #56 in the table. The answer is $(4 + 3\pi)/12$.

(c) $I = x\int_0^{1-x} dz = x(1-x)$. $H = \int_{y^2}^1 (x - x^2)\,dx = 1/6 - (y^4/2) + (y^6/3)$.
$G = \int_0^1 (1/6 - y^4/2 + y^6/3)\,dy = 4/35$.

(d) $I = \int_0^{1-y^2} z\,dz = 1/2\,(1-y^2)^2$. $H = 1/2\int_0^{1-x}(1 - 2y^2 + y^4)\,dy$
$= 1/2[(1-x) - 2/3(1-x)^3 + 1/5(1-x)^5]$. $G = 1/2\int_0^1[(1-x) - 2/3(1-x)^3$
$+ 1/5(1-x)^5]\,dx = 11/60$.

(e) $I = \int_0^c (x^2 + y^2 + z^2)\,dz = c(x^2 + y^2) + c^3/3$. $H = \int_0^b [c(x^2 + y^2) + c^3/3]\,dy$
$= bcx^2 + cb^3/3 + bc^3/3$. $G = \int_0^a (bcx^2 + cb^3/3 + bc^3/3)\,dx = bca^3/3 + acb^3/3$
$+ abc^3/3 = (abc/3)(a^2 + b^2 + c^2)$.

(f) $I = \int_0^{x/3} [x/(x^2 + y^2)]\,dy = \tan^{-1} \tfrac{1}{3}$. $H = (\tan^{-1} \tfrac{1}{3}) \int_0^z dx = z \tan^{-1} \tfrac{1}{3}$.
$G = \tfrac{3}{2} \tan^{-1} \tfrac{1}{3}$.

(g) $I = xy \int_0^{2-x} z\,dz = \tfrac{1}{2}xy(2 - x)^2$. $H = \tfrac{1}{2}x(2 - x)^2 \int_0^{1-x} dy = \tfrac{1}{2}x(1 - x)(2 - x)^2$.
$G = \tfrac{1}{2} \int_0^1 x(1 - x)(2 - x)^2\,dx = \tfrac{13}{240}$.

2. By (40), $M = \int_0^5 \int_0^{\sqrt{25-y^2}} \int_0^{\sqrt{25-x^2-y^2}} z\,dz\,dx\,dy = \tfrac{1}{2} \int_0^5 \int_0^{\sqrt{25-y^2}} (25 - x^2 - y^2)\,dx\,dy$
$= \tfrac{1}{3} \int_0^5 (25 - y^2)^{3/2}\,dy$. With the aid of integral formula #31, we find
$M = 625\pi/16$.

3. By (36) and by the same reasoning that led to (40), we find
$M = \int_0^5 \int_0^{\sqrt{25-x^2}} \int_0^{\sqrt{25-x^2-z^2}} z\,dy\,dz\,dx = \int_0^5 \int_0^{\sqrt{25-x^2}} z\sqrt{25 - x^2 - z^2}\,dz\,dx$
$= -\tfrac{1}{2} \int_0^5 \tfrac{2}{3}[25 - x^2 - z^2]^{3/2} \Big|_{z=0}^{z=\sqrt{25-x^2}} dx = \tfrac{1}{3} \int_0^5 [25 - x^2]^{3/2}\,dx$. As in Exercise
2, we obtain $M = 625\pi/16$.

4. (a) See Fig. 24. Since the problem has complete symmetry in x, y, z, we arbitrarily decide to integrate first with respect to z. The surfaces determining the boundary with respect to the z-values are the xy-plane below and the plane $x/a + y/b + z/c = 1$ above. Thus for the z-integration the lower and upper limits are given by $z = 0$ and
$z = c(1 - x/a - y/b)$ respectively. Thus $V = \int_A \int F(x, y)\,dy\,dx$ where A is shown in Fig. 24 and where $F(x, y) = \int_0^{c(1-x/a-y/b)} dz$. The xy-domain A in the xy-plane is bounded by the coordinate axes and the trace of $x/a + y/b + z/c = 1$ in the xy-plane which is $x/a + y/b = 1$, $z = 0$. Thus we obtain $V = \int_0^a \int_0^{b(1-x/a)} \int_0^{c(1-x/a-y/c)} dz\,dy\,dx = abc/6$.

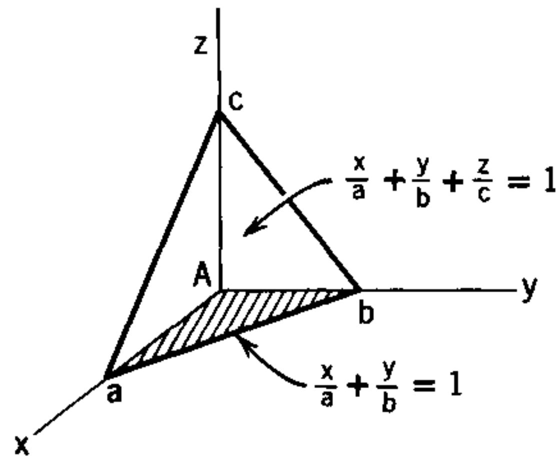


Fig. 24

(b) See Fig. 25. Here it is most convenient to integrate first with respect to x. The boundary surfaces with respect to x are $x = 0$ above and the paraboloidal surface $x = y^2 + z^2 - 1$ below. The yz-domain A is the trace of the paraboloid in the yz-plane which is $y^2 + z^2 = 1$, $x = 0$. Thus we obtain $V = \int_{-1}^{1} \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{y^2+z^2-1}^{0} dx\,dy\,dz = 4 \int_0^1 \int_0^{\sqrt{1-z^2}} \int_{y^2+z^2-1}^{0} dx\,dy\,dz.$ Integral formulas #31, 28 and 26 are helpful in obtaining the result $V = \pi/2$.

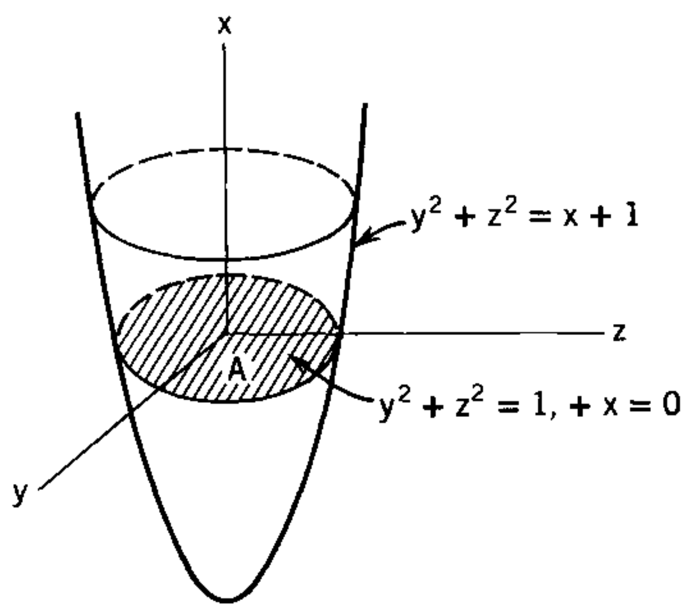


Fig. 25

(c) See Fig. 26. We integrate first with respect to z. The bounding surfaces with respect to z are $z = 0$ below and the two surfaces $z = 1 - x^2$, $z = 1 - y^2$ above. For a given z, $1 - x^2 = 1 - y^2$, so that the two surfaces meet where $x = y$ in the first octant. For any given (x, y) in the xy-domain, we must integrate from $z = 0$ to the greater of the two z-values given by $1 - x^2$, $1 - y^2$. Clearly for $|x| < |y|$ the first of these is the greater and vice versa for $|x| > |y|$. Thus for the area designated A_1 in Fig. 26 which is bounded by $y = 0$, $x = 1$, $x = y$ in the xy-plane, we must integrate $z = 0$ to $z = 1 - x^2$. Since by symmetry the volume over A_2 is obviously the same as over A_1, we have

$$V = 2\int_{A_1}\int\int_0^{1-x^2} dz\,dy\,dx = 2\int_0^1\int_0^x\int_0^{1-x^2} dz\,dy\,dx = 1/2.$$

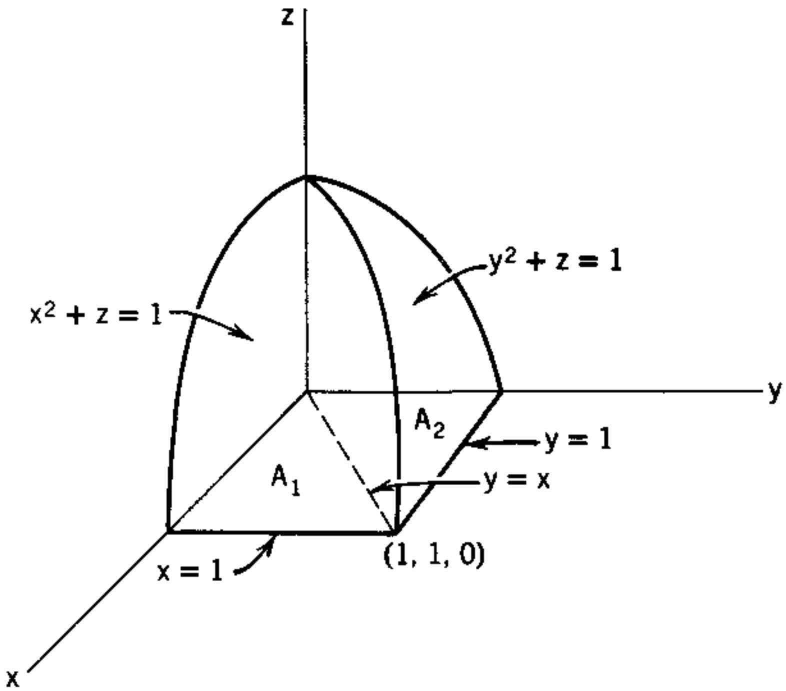


Fig. 26

(d) See Fig. 27. The bounding surfaces with respect to z are $z = 0$, $z = mx$. The xy-domain A is half of the circle $x^2 + y^2 = r^2$. Since the same volume lies over both quarter-circles, we have

$$V = 2\int_0^r\int_0^{\sqrt{r^2-x^2}}\int_0^{mx} dz\,dy\,dx = 2m\int_0^r \sqrt{r^2 - x^2}\,x\,dx = 2mr^3/3.$$

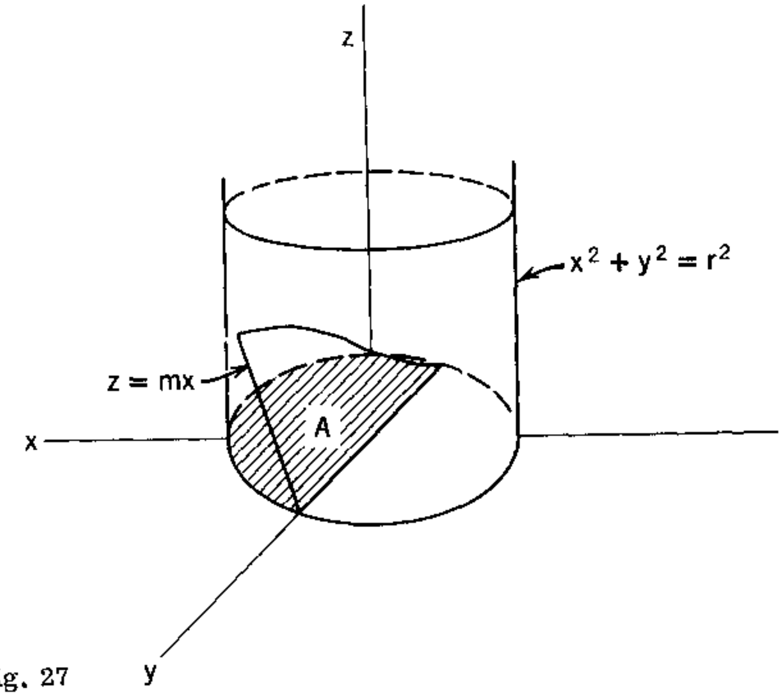


Fig. 27

(e) The volume lies in the portion of the cylinder below the plane $y + z = 4$ and above the xy-plane. The domain of x and y is the circle $x^2 + y^2 = 4$. Hence $V = \int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_0^{4-y} dz\,dx\,dy = 2\int_{-2}^{2}\int_0^{\sqrt{4-y^2}}(4-y)\,dx\,dy$ $= \int_{-2}^{2}(4x - yx)\Big|_0^{\sqrt{4-y^2}}dy = \int_{-2}^{2}(4\sqrt{4-y^2} - y\sqrt{4-y^2})\,dy$. The first integral is handled by integral formula #26 and the second by letting $u = 4 - y^2$. The answer is 16π.

(f) The parabolic cylinder $y^2 = ax$ cuts the paraboloid $y^2 + z^2 = 4ax$. The volume (Fig. 28) in question lies below the paraboloid and "inside" the cylinder and between $x = 0$ and $x = 3a$. Hence if we consider what lies in the first octant and multiply by 4 we have

$V = 4\int_0^{3a}\int_0^{\sqrt{ax}}\int_0^{\sqrt{4ax-y^2}} dz\,dy\,dx = 4\int_0^{3a}\int_0^{\sqrt{ax}}\sqrt{4ax - y^2}\,dy\,dx$. To integrate use integral formula #26 with the a^2 replaced by 4ax. Then

$V = 2\int_0^{3a}[y\sqrt{4ax - y^2} + 4ax\sin^{-1}(y/\sqrt{4ax})]\Big|_0^{\sqrt{ax}}dx$

$= 2\int_0^{3a}(\sqrt{ax}\sqrt{3ax} + 4ax\sin^{-1}\tfrac{1}{2})\,dx = 2\sqrt{3}a\int_0^{3a}x\,dx + \tfrac{4}{3}\pi a\int_0^{3a}x\,dx$

$= 9\sqrt{3}a^3 + 6\pi a^3$.

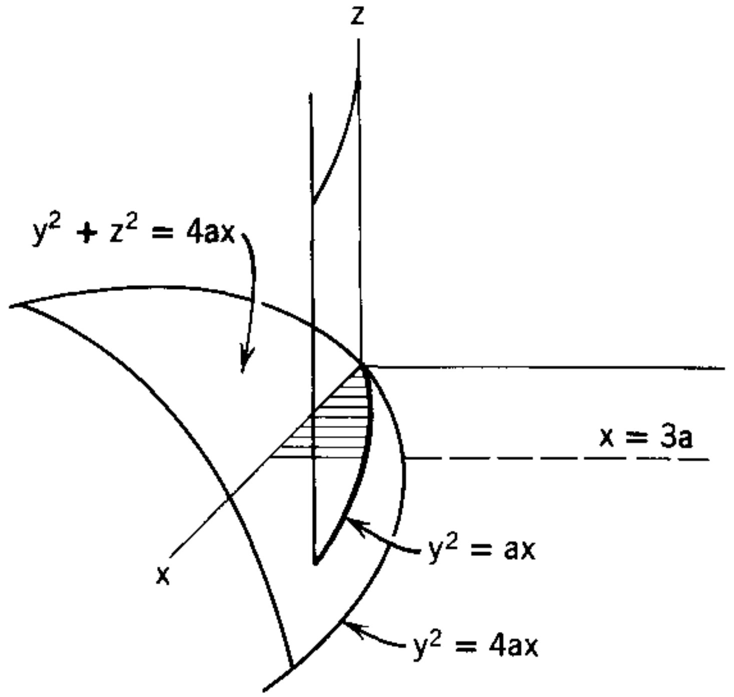


Fig. 28

(g) The area common to $y = x^2$ and $x = y^2$ is shown in Fig. 29. Since the surface $z = 12 + y - x^2$ lies above this area, we find
$V = \int_0^1 \int_{\sqrt{x}}^{x^2} \int_0^{12+y-x^2} dz\,dy\,dx = {}^{569}/_{140}$.

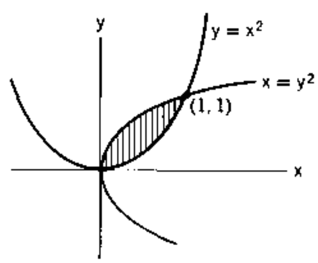


Fig. 29

(h) The bounding surfaces with respect to z are $z = x^2 + 3y^2$ below and $z = 8 - x^2 - y^2$ above. These surfaces intersect in the ellipse $x^2 + 2y^2 = 4$ which is shown shaded in Fig. 30. Then using the symmetry of the body, we have $V = 4\int_0^2 \int_0^{\sqrt{(4-x^2)/2}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx$
$= (8/\sqrt{2}) \int_0^2 \{4\sqrt{4-x^2} - x^2\sqrt{4-x^2} - {}^1/_3 (4-x^2)^{3/2}\}\,dx$
$= (16/3\sqrt{2}) \int_0^2 (4-x^2)^{3/2}\,dx$. Using integral formula #31 we find $V = 8\pi\sqrt{2}$.

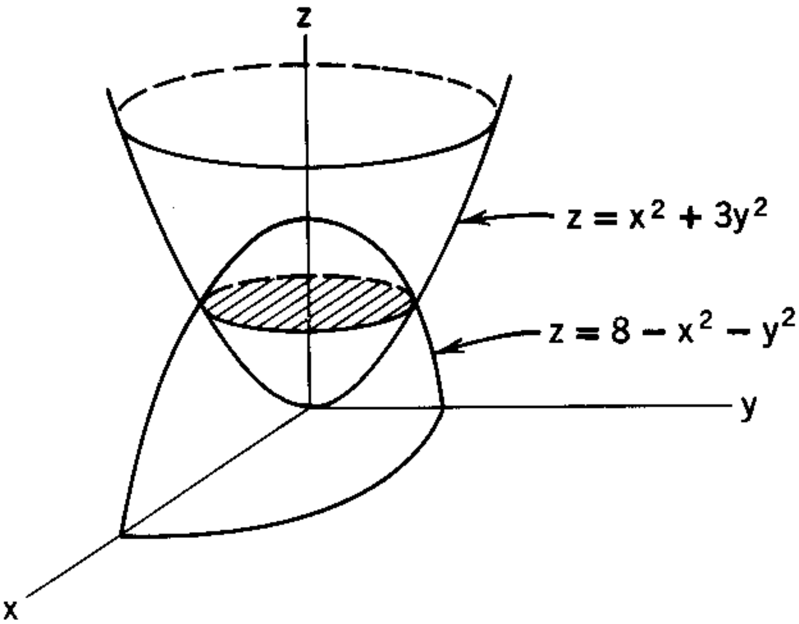


Fig. 30

(i) Noting that $z = 2x^2 + y^2$ and $z = 4 - y^2$ (see Fig. 31) intersect along the circle $x^2 + y^2 = 2$, we obtain as in (h),

$$V = 4\int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}}\int_{2x^2+y^2}^{4-y^2} dz\,dy\,dx = 4\pi.$$

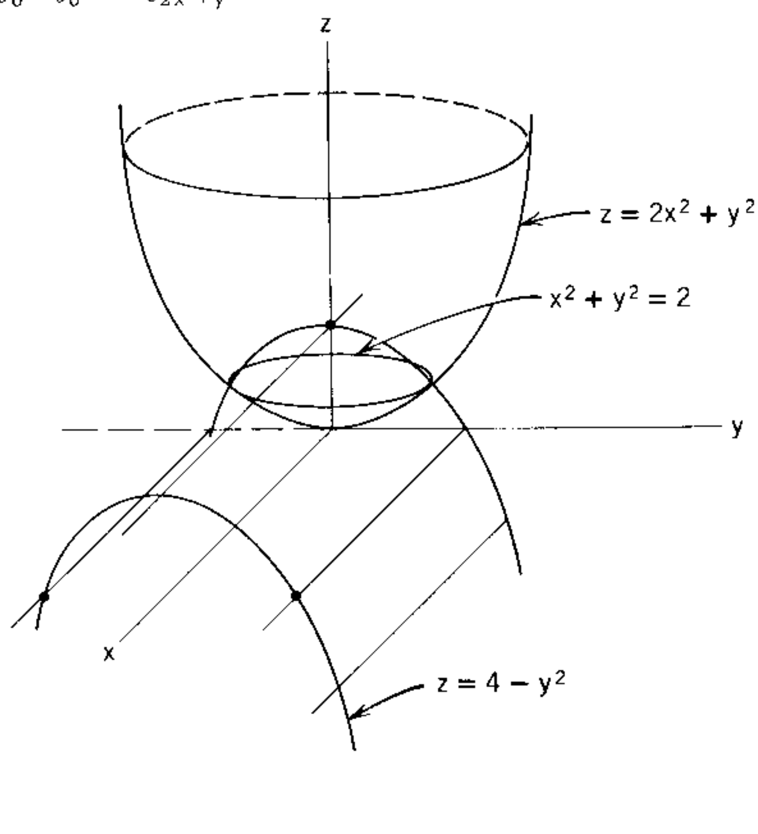


Fig. 31

5. As in the example of the text, the masses of the two hemispheres are given respectively by $4\int_0^5\int_0^{\sqrt{25-x^2}}\int_0^{\sqrt{25-x^2-y^2}} z\,dz\,dy\,dx$ and $4\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2-y^2}} z\,dz\,dy\,dx$. The mass of the shell is the difference of the masses. Evaluating the integrals as in Exercise 2, we obtain the result that the mass is 136π. Thus the weight is $136\pi g$.
6. See Fig. 32. The bounding surfaces with respect to z are $z = x + y$ below and $z = 1$ above. The xy domain is the triangle shown in the $z = 1$ plane. Since the density is $k(1 - z)$ we find that the mass is $k\int_0^1\int_0^{1-x}\int_{x+y}^1 (1-z)\,dz\,dy\,dx = k/24$.

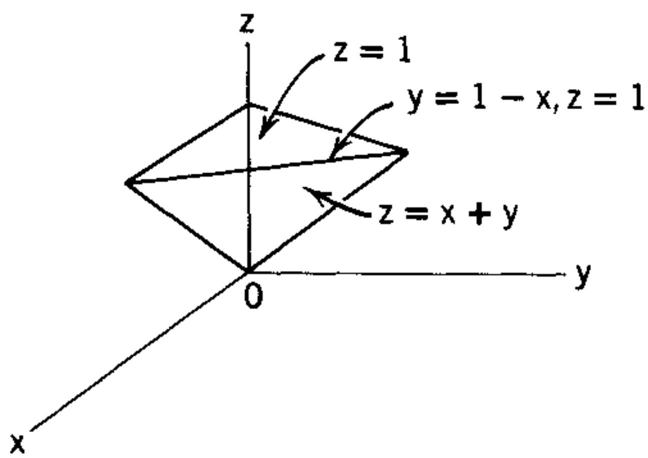


Fig. 32

7. See Fig. 33. In the usual way we obtain fig. w-60 $M = 2k\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^h yz\,dz\,dy\,dx = kh^2a^3/3$.

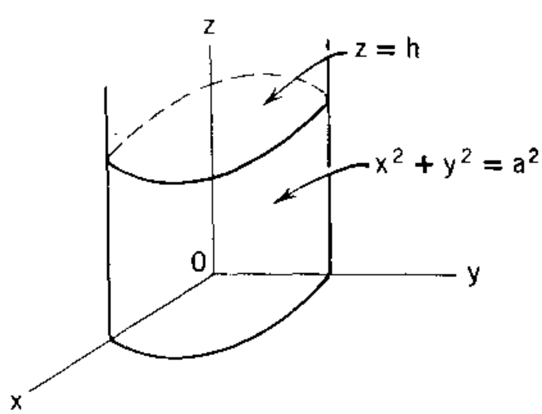


Fig. 33

8. The magnitude of the force of gravitation exerted by a mass element of density ρ on a mass m is $(Gm\rho/d^2)\,dx\,dy\,dz$, where d is the distance between the mass element and the mass m. If the mass element is located at the point (x, y, z) and m is at the origin we have $d^2 = x^2 + y^2 + z^2$. To obtain the x-component of the force, we must multiply by the first direction cosine of the line segment between the origin and the point (x, y, z). This is x/d; thus the x-component of the force is

$$\int_0^1\int_0^1\int_0^1 [Gmkyz/(x^2 + y^2 + z^2)] \cdot (x/\sqrt{x^2 + y^2 + z^2})\,dx\,dy\,dz$$

$= Gkm \int_0^1\int_0^1\int_0^1 [xyz/(x^2 + y^2 + z^2)^{3/2}]\,dx\,dy\,dz$. The integrations are readily executed. For example, to integrate with respect to x, let $u = x^2 + y^2 + z^2$. Then $du/dx = 2x$. y and z are constants. The answer is $Gkm(2\sqrt{2} - 1 - \sqrt{3})$.

9. The bounding surfaces with respect to z are the z = 0 plane below and the cone $z^2 = x^2 + y^2$ above. The xy-domain is bounded by the x-axis and the curve $y = 1 + \sqrt{1 - x^2}$ as x ranges from 0 to 1. Since the curve is the upper part of the circle with center (0, 1) and radius 1 in the xy-plane (it can be put into the form $x^2 + (y - 1)^2 = 1$) we conclude that the volume in question is the one lying above the domain in the xy-plane shown shaded in Fig. 34 and below the cone $z^2 = x^2 + y^2$.

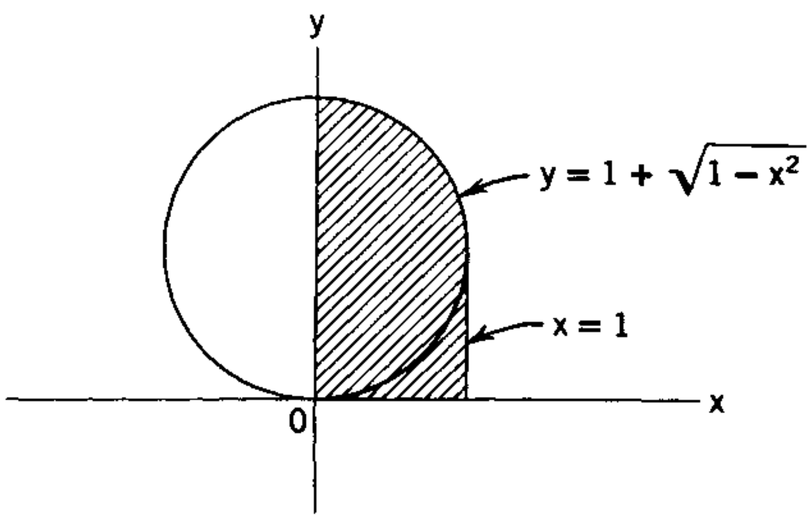


Fig. 34

10. The mass is given by $\int_0^1\int_0^{1+\sqrt{1-x^2}}\int_0^{\sqrt{x^2+y^2}} \rho\,(x, y, z)\,dz\,dy\,dx$. Here $\rho = k/\sqrt{x^2 + y^2 + z^2}$. Using integral formula #38 for the z-integration (with $a^2 = x^2 + y^2$) we find $\int_0^{\sqrt{x^2+y^2}} \rho\,dz = k\{\log(z + \sqrt{x^2 + y^2 + z^2})\Big|_{z=0}^{z=\sqrt{x^2+y^2}}\}$

$= k\log(1 + \sqrt{2})$ because terms cancel. Thus the mass is

$k\log(1 + \sqrt{2})\int_0^1\int_0^{1+\sqrt{1-x^2}} dy\,dx = k[\log(1 + \sqrt{2})](1 + \pi/4)$.

11. The surface $z = xy/a$ is the hyperbolic paraboloid shown in Fig. 21-23 but turned $45°$ so that the figure is symmetric about the plane $x = y$ instead of the y-axis. The surface $x + y + z = a$ is of course a plane which cuts the former surface in the first octant. The bounding surfaces with respect to z are $z = 0$ below and the smaller of $z = xy/a$, $z = a - x - y$ above. The latter two surfaces are at equal heights when $xy/a = a - x - y$. Thus in the region marked H in Fig. 35 we integrate up to $z = xy/a$, while in the region marked P we integrate up to $z = a - x - y$. Writing the curve dividing the regions H and P in the form $y = a(a - x)/(a + x)$ we arrive at the formula for the volume:

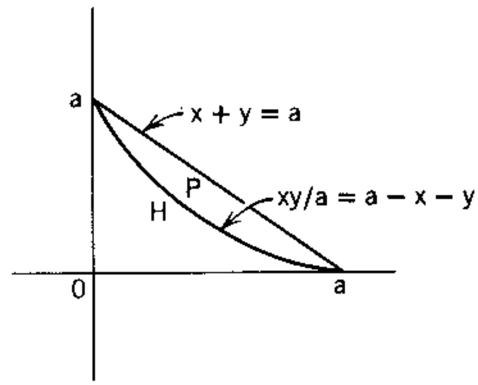


Fig. 35

$V = \int_0^a \int_a^{a(a-x)/(a+x)} \int_0^{xy/a} dz\,dy\,dx + \int_0^a \int_{a[(a-x)/(a+x)]}^{a-x} \int_0^{a-x-y} dz\,dy\,dx$. After grouping terms the result of the first two integrations is $V = (a/2) \int_0^a [x(a - x)^2/(a + x)^2]\,dx$ $+ \frac{1}{2} \int_0^a [x^3(a - x)^2/(a + x)^2]\,dx = \frac{1}{2} \int_0^a [x(a - x)^2/(a + x)]\,dx$. The substitution $u = x + a$ leads to $V = a^3(17 - 24 \log 2)/12$.

12. The volume lies between $z = 0$ and $z = c\sqrt{1 - x^2/a^2 - y^2/b^2}$ above the first quadrant of the ellipse $x^2/a^2 + y^2/b^2 = 1$. Hence $V = \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \int_0^{c\sqrt{1-x^2/a^2-y^2/b^2}} dz\,dy\,dx$. Use of integral formula #26 in the second integration leads to $V = \pi abc/6$.

13. The figure is similar to that shown in Fig. 30 for Exercise 4(h). Thus $V = \int_0^1 \int_0^{\sqrt{8-8x^2}} \int_{3x^2+(1/4)y^2}^{4-x^2-(1/4)y^2} dz\,dy\,dx$. The answer is $4\pi\sqrt{2} = 17.17$ approx.

14. The domain of integration lies above the shaded area in Fig. 36 and under the cylinder $y^2 + z^2 = 4$. The equations $x + y = 2$ and $x + 2y = 6$ represent planes which rise vertically and cut the cylinder. The figure shows the traces of these planes in the plane $z = 0$. Then $V = \int_0^2 \int_{2-y}^{6-2y} \int_0^{\sqrt{4-y^2}} z\,dz\,dx\,dy$. The integration is straightforward and the answer is $^{26}/_3$.

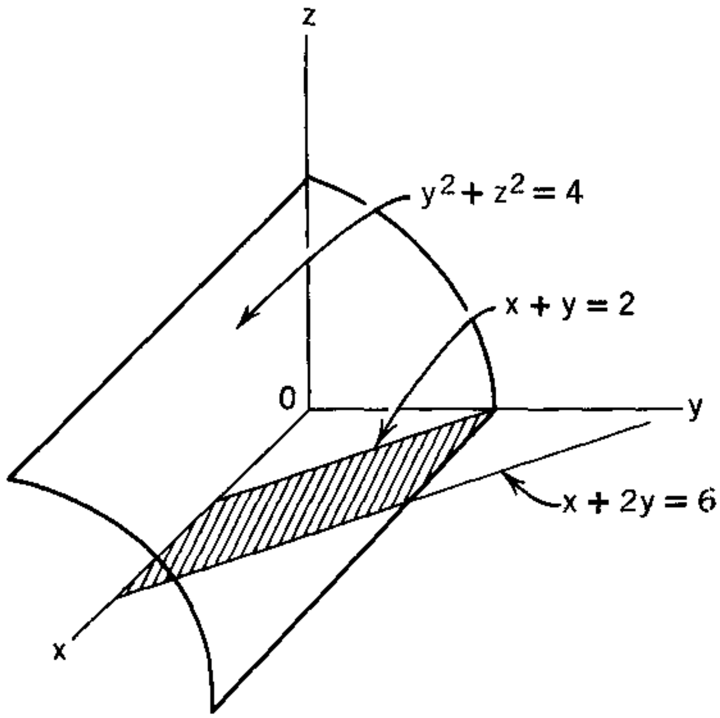


Fig. 36

15. The bounding surfaces with respect to z are the plane $z = 4$ above and the paraboloid $z = \frac{1}{4}(x^2 + y^2)$ below. The xy-domain is bounded by the curves $y = 0$ and $y = \sqrt{16 - x^2}$ as x ranges from 0 to 4. This domain is thus the quarter of the circle $x^2 + y^2 = 16$ lying in the first quadrant. Thus the volume lies between the plane $z = 4$ and the paraboloid $z = \frac{1}{4}(x^2 + y^2)$ and above this quarter-circle.

CHAPTER 23, SECTION 7

1. (a) After two integrations we obtain $\int_0^{\pi/4} (\frac{1}{2}\sec^2\theta - \sec\theta + \frac{1}{2})\,d\theta$. Recalling that $\int \sec^2\theta\,d\theta = \tan\theta + C$ and using integral formula #9, we obtain the result $[(\pi + 4)/8] - \log(\sqrt{2} + 1)$.
   (b) The integrations are simple and the answer is $2(1 - \cos 2)/3$.
   (c) Use integral formula #27 to obtain $256\pi/5$.
2. To take best advantage of the solution in the text we locate the unit mass at

the origin and place the cylinder with its axis along the z-axis with its lower base c units above the xy-plane. Then the theory of the text applies at once except that the z-end values are c and c + h. Hence the integral for the gravitational attraction is $\int_a^b \int_c^{c+h} \int_0^{2\pi} GDz\rho \, d\theta \, dz \, d\rho/(\rho^2 + z^2)^{3/2}$ and straightforward evaluation gives the result in the text. We can obtain the same answer by supposing that we have a longer shell of height h + c and a shell of height c. The latter's attraction opposes the former's. Both shells have the unit mass at the center of the lower base. Then we take the text's result with h replaced by h + c and subtract the text's result with h replaced by c.

3. We may represent the cone by $z^2 = \frac{9}{4}(x^2 + y^2)$ or in polar coordinates by $z = \frac{3}{2}\rho$. The bounding surfaces with respect to z are the cone below and the plane $z = 3$ above. Thus $V = \int_0^{2\pi} \int_0^2 \int_{3\rho/2}^3 \rho \, dz \, d\rho \, d\theta = 4\pi$.
4. The paraboloid may be written in the form $z = \frac{1}{2}\rho^2$. Thus $V = \int_0^{2\pi} \int_0^2 \int_{\rho^2/2}^2 \rho \, dz \, d\rho \, d\theta = 4\pi$.
5. The bounding surfaces with respect to z are $z = 0$ and $z = h$. The $\rho\theta$-domain may be described by $\rho = a$, $0 \le \theta \le \pi$. Thus the mass is $\int_0^h \int_0^\pi \int_0^a Dz\rho \sin\theta \cdot \rho \, d\rho \, d\theta \, dz = Da^3h^2/3$.
6. We first note that by symmetry only a vertical force acts and that the angle ψ in Fig. 23-27 is given by $\cos\psi = z/r = z/\sqrt{\rho^2 + z^2}$. Using the suggestion and proceeding as in the text example we obtain the expression $\int_0^{2\pi} \int_0^a \int_{h\rho/a}^h [GDz/(\rho^2 + z^2)^{3/2}] \rho \, dz \, d\rho \, d\theta = 2\pi GDh\{1 - h/\sqrt{a^2 + h^2}\}$.
7. Once again by symmetry only a vertical force acts. The bounding surfaces are $z = \sqrt{2 - \rho^2}$ and $z = \rho^2$. These surfaces intersect along $2 - \rho^2 = \rho^4$ or along $(\rho^2 + 2)(\rho^2 - 1) = 0$. The real intersection is the circle $\rho = 1$. As in the example in the text the vertical force is $G\Delta Mz/r^3 = G\Delta Mz/(\rho^2 + z^2)^{3/2}$. $\Delta M = D\Delta V$ where D is the density. Then the integral is $\int_0^{2\pi} \int_0^1 \int_{\rho^2}^{\sqrt{2-\rho^2}} [GDz/(\rho^2 + z^2)^{3/2}] \rho \, dz \, d\rho \, d\theta$. The integration is straightforward and the answer is $\pi GD/\sqrt{2}$.
8. The cylinder has its center on the x-axis at $x = a$ (or $\rho = a$, $\theta = 0$). The cone rises from the origin and cuts through the cylinder. To obtain the x-component of the force we must multiply the magnitude of the force by the first direction cosine of the line from the origin (pole) to the point (ρ, θ, z). This direction cosine is $x/\sqrt{x^2 + y^2 + z^2} = \rho\cos\theta/\sqrt{\rho^2 + z^2}$. Thus the x-component of the force is $GD\rho\cos\theta/(\rho^2 + z^2)^{3/2}$, where D (or k) is the density. The domain of integration is from $z = 0$ to $z = c\rho$ and over the circle $\rho = 2a\cos\theta$. Hence the total x-component is $\int_{-\pi/2}^{\pi/2} \int_0^{2a\cos\theta} \int_0^{c\rho} [GD\rho\cos\theta/(\rho^2 + z^2)^{3/2}] \rho \, dz \, d\rho \, d\theta$. Use integral formula #45 to do the first integration. Ans. $\pi GDca/\sqrt{1 + c^2}$.

9. The cylinder $\rho = a\cos 2\theta$ is parallel to the z-axis and cuts the xy-plane (the $\rho\theta$-plane) in a four-leaved rose. The plane $z = 4a + \rho\cos\theta$ ($=4a + x$) cuts the entire cylinder above the xy-plane. Hence

$V = \int_0^{2\pi}\int_0^{a\cos 2\theta}\int_0^{4a+\rho\cos\theta} \rho\, dz\, d\rho\, d\theta$. The evaluation of the triple integration is straightforward but lengthy because terms involving $\cos^2 2\theta$ and $\cos^3 2\theta\cos\theta$ occur in the integration with respect to θ. One must convert these to $\sin\theta$ and $\cos\theta$ by using $\cos 2\theta = \cos^2\theta - \sin^2\theta$. The answer is $2\pi a^3$.

10. As in the text example, only the vertical component of the gravitational force acts and this is given by $GMz\rho\, d\rho\, dz\, d\theta/(\rho^2 + z^2)^{3/2}$. Here M is the equivalent of the k in the text. The volume lies over the domain $\rho^2 = 4a^2$ in the $\rho\theta$-plane. Hence the integral is

$GM\int_0^{2\pi}\int_0^{2a}\int_{\rho^2/4a}^{a}[\rho z/(\rho^2 + z^2)^{3/2}]\,dz\, d\rho\, d\theta$. The integration is straightforward. The integration with respect to ρ requires the use of integral formula #38.

11. As in the text example, only a vertical force acts and this is $GM\cos\psi/r^2$, where M is the density. Here, however, the unit mass is located at $\rho = 0$, $\theta = 0$, $z = 2a$. Hence $\cos\psi = (2a - z)/r$ and $r = \sqrt{\rho^2 + (2a - z)^2}$. z varies from the paraboloid $z = \rho^2/4a$ to $z = 2a$. The domain of ρ and θ values is the circle $\rho^2 = 8a^2$ or $\rho = 2\sqrt{2}a$. Hence the attraction is given by

$\int_0^{2\pi}\int_0^{2\sqrt{2}a}\int_{\rho^2/4a}^{2a}\{GM(2a - z)/[\rho^2 + (2a - z)^2]^{3/2}\}\,\rho\, dz\, d\rho\, d\theta$. The integration with respect to z is accomplished by letting $u = \rho^2 + (2a - z)^2$. After substitution of the z-end values and simplification we obtain

$GM\int_0^{2\pi}\int_0^{2\sqrt{2}a}(1 - 4a\rho/\sqrt{64a^4 + \rho^4})\,d\rho\, d\theta$. To achieve the integration let $\rho^2 = 8a\tan\phi$ in the second term. This leads for this term to $\int\sec\phi\, d\phi$ which is a standard form. The answer is in the text.

12. The value lies under the paraboloid which opens downward. The correct integral is $\int_0^{2\pi}\int_0^{3}\int_0^{9-\rho^2}\rho^2\cdot\rho\, dz\, d\rho\, d\theta$. The answer is in the text.

13. The cylinder cuts the $\rho\theta$-plane in a circle whose center is on the x-axis (polar axis) at $x = 2$. The sphere cuts through the entire cylinder above the xy-plane ($\rho\theta$-plane). Hence the volume lies between $z = 0$ and $z = \sqrt{16 - \rho^2}$ and over the $\rho\theta$-domain of the circle $\rho = 4\cos\theta$. Then

$V = 2\int_0^{\pi/2}\int_0^{4\cos\theta}\int_0^{\sqrt{16-\rho^2}}\rho\, dz\, d\rho\, d\theta$. The integration is straightforward. $V = {}^{64}\!/_9(3\pi - 4)$. Note that if one takes θ end values from $-\pi/2$ to $\pi/2$ instead of multiplying by 2 one gets ${}^{64}\!/_9(3\pi)$ but one gets the correct value if one also takes 0 and π as the end values for θ. The discrepancy comes up because in going from $-\pi/2$ to $\pi/2$ one encounters a negative "area" cancelling a positive one.

CHAPTER 23, SECTION 8, FIRST SET

1. (a) Sphere, center at origin and radius 4.
   (b) Upward facing cone, vertex at origin and semi-vertex angle of 30°.
   (c) Downward facing cone, vertex at origin and semi-vertex angle of 60°.
   (d) Plane parallel to and one unit above the xy-plane ($\phi = 90°$ plane).
   (e) Half plane perpendicular to xy-plane ($\phi = 90°$plane) and making an angle of 30° with the x-axis ($\theta = 0$ half plane).
   (f) Half plane perpendicular to the $\phi = 90°$plane and making an angle of 210° with the $\theta = 0$ half plane.
   (g) Cylinder, axis is z-axis and radius is 2.
   (h) Sphere, center at $x = y = 0$, $z = a/2$ ($\rho = a/2$, $\phi = 0$, $\theta = 0$) and radius $a/2$.

CHAPTER 23, SECTION 8, SECOND SET

1. (a) The result of the first integration is $\sec\phi \sin 2\phi = 2 \sin\phi$. The remaining integrations are readily performed.
   (b) After two elementary integrations we obtain $3 \cos(\pi/4) - 3 \cos(\operatorname{arc\,tan} 2)$. But $\cos(\operatorname{arc\,tan} 2) = 1/\sqrt{5}$. Then the final result is $(3\pi/2)[(1/\sqrt{2}) - (1/\sqrt{5})]$.
   (c) The integrations are straightforward.
2. Take the center of the sphere at the origin. Then the equation of the sphere is $\rho = 6$ and that of the cone is $\phi = 30°$. Then for fixed θ and ϕ, ρ goes from 0 to 6. ϕ varies from 0 to $\pi/6$ and θ from 0 to 2π. Thus
   $V = \int_0^{2\pi}\int_0^{\pi/6}\int_0^{6} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$. The final result is $144\pi(1 - \sqrt{3}/2)$.
3. The equation $\rho = 2a \cos\phi$ represents in the $\theta = 0$ plane, a semicircle with center at $\rho = a$ and $\phi = 0$. Rotation of this semicircle around the $\phi = 0$ axis gives the sphere. The sphere passes through the origin and the cone $\phi = \pi/4$ has its vertex there. The cone lies inside the sphere until it cuts through at the top. Hence except for the end values of ρ we have the same situation as in Exercise 2. $V = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2a\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$. The integrations are straightforward.
4. The mass is the integral of the density over the volume of the sphere. Hence taking the center of the sphere at the origin,
   $M = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} (k/\rho^2)\, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = 4\pi kR.$
5. Because of the symmetry of the cone only the vertical component of the force of gravity is effective. If we take an element of volume dV in the cone, the force it exerts on unit mass at the vertex is Gk/ρ^2 where k is the density. The vertical component downward (the cone extends upward

from the origin) is $Gk\cos\phi/\rho^2$. To describe the volume we note that for fixed θ and ϕ, ρ varies from the origin to the base of the cone where $\rho\cos\phi = h$. Hence ρ varies from 0 to $h/\cos\phi$. Then

$$F = \int_0^{2\pi}\int_0^{\alpha}\int_0^{h/\cos\phi} (Gk\cos\phi/\rho^2)\,\rho^2\sin\phi\; d\rho\, d\phi\, d\theta.$$

The integration is straightforward and the result is in the text.

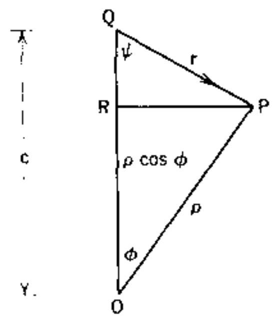


Fig. 37

6. The solution follows the text beginning on p. 298 with the following changes. Since the unit mass is external to the shell now $c > R_2$. In Figure 23-33 this means that Q lies above R. The vertical component of the attractive force (cf. (53)) is still $GM\Delta V\cos\psi/r^2$ where ψ is shown in Figure 37. Now $\cos\psi = (c - \rho\cos\phi)/r$. Hence the vertical component is $GM\Delta V(c - \rho\cos\phi)/r^3$. Proceeding as in the text we come, in place of (58), to

$\int_{R_1}^{R_2}\int_0^{\pi}\int_0^{2\pi} [GM(c - \rho\cos\phi)/(\rho^2 + c^2 - 2\rho c\cos\phi)^{3/2}]\,\rho^2\sin\phi\; d\theta\, d\phi\, d\rho$. We integrate with respect to ϕ first and, as in the text, break up the integral into two integrals. Because we have $c - \rho\cos\phi$ in place of $\rho\cos\phi - c$ in the text, we get in place of (61), $[1/\rho(\rho^2 + c^2 - 2\rho c\cos\phi)^{1/2}]\Big|_0^{\pi}$. Again, the radical is positive. When we substitute π for ϕ the radical yields $\rho + c$. When we substitute 0 for ϕ, this time we must take $c - \rho$ to keep the radical positive. Then we get $1/\rho(\rho + c) - 1/\rho(c - \rho) = -2/(c^2 - \rho^2)$. To evaluate the second integral we proceed as in the text beginning with (63), except that we have a minus sign in front of the integral. We come to (64), except for the minus signs in front of the main terms. Again when we substitute π and 0 for ϕ we must remember that $c > \rho$; in place of (65) we get $-2\rho^2/c^2(c^2 - \rho^2)$. We must now add this result to the previous one, that is, $-[2/(c^2 - \rho^2)] - 2\rho^2/c^2(c^2 - \rho^2) = -2/c^2$. Then the integral (58) becomes $-\int_{R_1}^{R_2}\int_0^{2\pi} (2GM\rho^2/c^2)\,d\rho\, d\theta = -(4\pi GM/3c^2)(R_2^3 - R_1^3)$. The volume of the shell is $4\pi(R_2^3 - R_1^3)/3$. Hence the result says that we take the mass of the shell divided by the square of the distance from the center to the unit mass. The minus sign in the result comes from failing to fix a sign for the direction of the gravitational force.

7. Starting with (59) and letting $u^2 = \rho^2 + c^2 - 2\rho c\cos\phi$ we have $u\,du/d\phi = \rho c\sin\phi\, d\phi$. Also to convert $\rho\cos\phi$ to u we have from the expression for u^2 that $\rho\cos\phi = (\rho^2 + c^2 - u^2)/2c$. Take u to be the positive root. Then when $\phi = \pi$, $u = \rho + c$ and when $\phi = 0$, $u = \rho - c$ (since $\rho > c$). In place of (59) we now have $(1/2\rho c^2)\int_{\rho-c}^{\rho+c} [(\rho^2 - c^2)/u^2]\,(du/d\phi)\,d\phi$
$= (1/2\rho c^2)\int_{\rho-c}^{\rho+c} [(\rho^2 - c^2)/u^2)]\,du = 0.$

8. (a) By the symmetry of the figure the force acts along the line determined by $\phi = 90°$ and $\theta = 90°$. If we take any element ΔV of the volume and if k is the density then the gravitational force acting on the unit mass is $Gk\Delta V/\rho^2$. The component in the direction of the line just described is $Gk\Delta V \cos(90 - \phi)\cos(90 - \theta)/\rho^2$. One way to see this is to take the component parallel to the $\phi = 90°$ plane, which is $Gk\Delta V \cos(90 - \phi)$, and then then the component of this along the line $\phi = 90°$, $\theta = \pi/2$. Then the total force is $\int_0^\pi \int_0^\pi \int_0^a (Gk/\rho^2)\sin^2\phi \sin\theta\, \rho^2\, d\rho\, d\phi\, d\theta$. This give $Gka\pi$.

(b) The mass of the hemisphere is $2\pi ka^3/3$. If this mass is concentrated at $\rho = \sqrt{2/3}a$, $\theta = \pi/2$, $\phi = \pi/2$, its distance from the origin is $\sqrt{2/3}a$. Hence the gravitational attraction it exerts at the origin is $(G2\pi ka^3/3)/(2a^2/3) = Gka\pi$.

9. $V = \int_0^{2\pi}\int_0^\pi \int_0^{2a\sin\phi} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$. Since the end values are constants we may reverse the order of ϕ and θ. This gives $(16\pi a^3/3)\int_0^\pi \sin^4\phi\, d\phi$.

One can integrate in various ways, in particular by using the reduction formula #72 in the Table of Integrals. The text's answer results.

10. The spheres intersect along $\cos\phi = 3/4$ or $\phi = \cos^{-1} 3/4$. A simple sketch shows that for $0 < \phi < \cos^{-1} 3/4$, ρ ranges between 0 and 3 whereas for $\cos^{-1} 3/4 < \phi < \pi/2$, ρ ranges between 0 and $4\cos\phi$. Then the volume is

$$\int_0^{2\pi}\int_0^{\arccos(3/4)}\int_0^3 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta + \int_0^{2\pi}\int_{\arccos(3/4)}^{\pi/2}\int_0^{4\cos\phi} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta.$$

Using the relations $\cos(\cos^{-1} 3/4) = 3/4$ and $\cos^4(\cos^{-1} 3/4) = (3/4)^4$, we obtain the result $63\pi/8$.

11. If we take an element ΔV of volume the gravitational force it exerts on unit mass at the origin is $Gk\Delta V/\rho^2$, where k is the density. However by the symmetry of the figure only the vertical component of the force acts. This is $Gk\Delta V \cos\phi/\rho^2$. The domain of integration must be broken up, as in Exercise 10, into two domains with the end values precisely the same as in Exercise 10. The integration is simpler because the ρ^2 factors cancel. The result is in the text.

12. The density $D = k/\sqrt{x^2 + y^2} = k/\sqrt{\rho^2 - z^2} = k/\sqrt{\rho^2 - \rho^2\cos^2\phi} = k/\rho \sin\phi$. The bounding surfaces with respect to ρ are $\rho = 1/\cos\phi$ below and $\rho = 2$ above. Since the plane intersects the sphere along $\cos\phi = 1/2$, that is, $\phi = 60° = \pi/3$, the mass is given by $\int_0^{2\pi}\int_0^{\pi/3}\int_{1/\cos\phi}^2 (k/\rho \sin\phi)\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$. After the first integration we obtain $(k/2)\int_0^{2\pi}\int_0^{\pi/3}(4 - \sec^2\phi)\, d\phi\, d\theta$. Since $d(\tan\phi)/d\phi = \sec^2\phi$, we find the answer $(\pi k/3)(4\pi - 3\sqrt{3})$.

13. It is convenient to take the cone as opening downward so that its equation is $\phi = 120°$. Then the larger solid cut out by the cone lies above or outside the cone. The gravitational force which any element of the solid exerts at the origin is $Gk\Delta V/\rho^2$ where k is the density. By the symmetry of the solid only the vertical component of the force is effective. This component

is $Gk\Delta V \cos\phi/\rho^2$. Then the force is

$F = \int_0^{2\pi}\int_0^{2\pi/3}\int_b^c (Gk/\rho^2)\cos\phi\ \rho^2 \sin\phi\ d\rho\ d\phi\ d\theta$. This gives $F = Gk\pi 3(c-b)/4$.

Now k, the density, is M/V, where M is the total mass of the solid in question and V is the volume. To obtain an answer in terms of M, we leave this as is. $V = \int_0^{2\pi}\int_0^{2\pi/3}\int_b^c \rho^2 \sin\phi\ d\rho\ d\phi\ d\theta = \pi(c^3 - b^3)$. Replace k in the value of F by M/V and we get $F = 3GM/4(b^2 + bc + c^2)$.

14. As in previous Exercise, only the vertical component of the gravitational force is effective and this is $GD\Delta V \cos\phi/\rho^2$ where D is the density. We are given that $D = k\rho$. Hence the component is $Gk\Delta V \cos\phi/\rho$. We integrate over the volume. Then the total force

$F = \int_0^{2\pi}\int_0^{\pi/4}\int_0^2 (Gk\cos\phi/\rho)\rho^2 \sin\phi\ d\rho\ d\phi\ d\theta = Gk\pi$. To put the answer in terms of the mass M of the solid we compute M. This is

$\int_0^{2\pi}\int_0^{\pi/4}\int_0^2 k\rho\,\rho^2 \sin\phi\ d\rho\ d\phi\ d\theta$. We find readily that $M = 4k\pi(2 - \sqrt{2})$. Take the value of k from this and substitute it in $F = Gk\pi$. Then $F = GM(2 + \sqrt{2})/8$.

15. Here $V = \int_0^{2\pi}\int_{\pi/4}^{\arctan 2}\int_0^{\sqrt{6}} (1/\rho)\rho^2 \sin\phi\ d\rho\ d\phi\ d\theta$. The integration is straightforward. We need only note that $\cos(\arctan 2) = 1/\sqrt{5}$. The answer is $6\pi[(\sqrt{2}/2) - (1/\sqrt{5})]$.

CHAPTER 23, SECTION 9

In this section we use the following formulas for the moment of inertia of a homogeneous body of mass M: $I = (M/V)\iiint r^2\,dV$, $I = (M/A)\int_A\int r^2\,dA$ (thin disc), $I = (M/\ell)\int r^2\,dx$ (thin rod). These formulas summarize the derivations of the text. The symbols V, A, ℓ respectively stand for the volume, area and length of the body and r stands for the perpendicular distance from a typical point in the body to the axis of rotation.

1. (a) We use the formula for thin rods, $I = (M/\ell)\int r^2\,dx$. Here the length ℓ is 2a. Placing the rod along the x-axis with the axis of rotation through the origin, we see that $r = x$ varies between $-a$ and a. Thus

   $I = (M/2a)\int_{-a}^{a} x^2\,dx = Ma^2/3$.

   (b) The situation is similar to (a). Placing the rod along the x-axis with the axis of rotation through the origin, we have

   $I = (M/2a)\int_0^{2a} x^2\,dx = 4Ma^2/3$.

2. (a) Use the formula for thin discs, place the axis of rotation through the origin and use plane polar coordinates to obtain

$I = (M/\pi a^2) \int_0^{2\pi} \int_0^a \rho^2 \cdot \rho \, d\rho \, d\theta = \frac{1}{2} Ma^2.$

(b) In the result of (a), M can be replaced by density times volume $= D\pi a^2 t$, where D is the density and t the thickness of the disc. If we now let M be the mass of the solid cylinder, then $D = M/\pi a^2 \ell$ and $D \cdot \pi a^2 \cdot t = Mt/\ell$. Thus by (a) the moment of inertia of a thin slice of thickness dx is $\frac{1}{2}(Ma^2/\ell)\,dx$ and the moment of inertia of the whole cylinder is $\int_0^{\ell} (Ma^2/2\ell)\,dx = \frac{1}{2} Ma^2$, where now M is the mass of the solid cylinder.

3. Use the formula for thin rods with dx = ds = arclength. Thus

$I = (M/2\pi a) \int a^2 \, ds = (M/2\pi a) \int_0^{2\pi} a^2 \cdot a \, d\theta = Ma^2.$

4. Let the origin of a rectangular system be at the center of the disc. Then

$I = (M/A) \int_A \int (x^2 + y^2)\,dx\,dy = (M/4ab) \int_{-b}^{b} \int_{-a}^{a} (x^2 + y^2)\,dx\,dy = (M/3)(a^2 + b^2).$

5. $I = (M/V) \int\int_V \int (x^2 + y^2)\,dx\,dy\,dz$. Using spherical coordinates and noting that $x^2 + y^2 = \rho^2 - z^2 = \rho^2 - \rho^2 \cos^2\phi = \rho^2 \sin^2\phi$ and using integral formula #72, we obtain $I = (M/4\pi a^3/3) \int_0^{2\pi} \int_0^{\pi} \int_0^a \rho^4 \sin^3\phi \, d\rho \, d\phi \, d\theta = \frac{2}{5} Ma^2.$

6. Place the disc in the xy-plane with center at the origin and sides of length 2a and 2b respectively parallel to the x- and y-axes. Then the axis of rotation is the y-axis and r = x. Thus $I = (M/A) \int_A \int x^2\,dx\,dy$
$= (M/4ab) \int_{-b}^{b} \int_{-a}^{a} x^2\,dx\,dy = \frac{1}{3} Ma^2.$

7. (a) See Fig. 38. If the mass of the lamina is M and its area A, then its moment of inertia about the x-axis is $I_x = (M/A) \int\int y^2\,dx\,dy$ and its moment of inertia about the y-axis is $I_y = (M/A) \int\int x^2\,dx\,dy$. The moment of inertia about the z-axis is $I_z = (M/A) \int\int \rho^2\,dx\,dy$
$= (M/A) \int\int (x^2 + y^2)\,dx\,dy = I_x + I_y.$

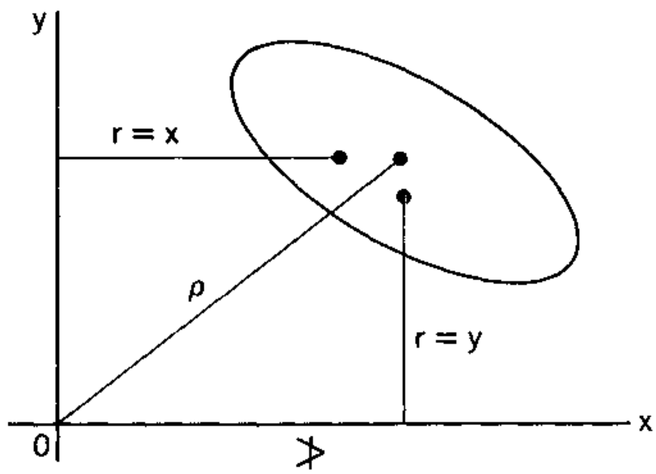


Fig. 38

(b) In 2(a) we found that the moment of inertia of circular disc about an axis perpendicular to its center was $\frac{1}{2}Ma^2$. Now the moments of inertia about all diameters are obviously equal; in particular if we choose 2 perpendicular diameters we have by (a) that $2I = \frac{1}{2}Ma^2$. Thus $I = \frac{1}{4}Ma^2$.

8. Using the equation of the cone in the form $\rho = az/h$, we have
$D\int_0^{2\pi}\int_0^h\int_0^{(a/h)z}\rho^3\,d\rho\,dz\,d\theta = \frac{1}{10}\pi Da^4h$.

9. We use spherical coordinates and take the y-axis ($\theta = \pi/2$, $\phi = \pi/2$) as the axis of rotation. Then $r^2 = x^2 + z^2 = \rho^2\cos^2\theta\,\sin^2\phi + \rho^2\cos^2\phi$. For a homogeneous body $(M/V) = D$, thus
$I = D\int_0^{2\pi}\int_0^{\pi/2}\int_b^a \rho^4(\cos^2\theta\,\sin^3\phi + \cos^2\phi\,\sin\phi)\,d\rho\,d\phi\,d\theta$. To integrate replace $\sin^3\phi$ by $(1 - \cos^2\phi)\sin\phi$. The rest is straightforward and leads to the text's answer.

10. We use cylindrical coordinates. Let the z-axis be the axis of rotation. Then $r^2 = x^2 + y^2 = \rho^2$. The equations of the bounding surfaces of the shell are $\rho^2 + z^2 = 16$, $\rho^2 + z^2 = 25$. Thus the bounding surfaces with respect to ρ are $\rho = \sqrt{16 - z^2}$, $\rho = \sqrt{25 - z^2}$. Hence $I = D\int_0^{2\pi}\int_1^3\int_{16-z^2}^{25-z^2}\rho^2\cdot\rho\,d\rho\,dz\,d\theta$
$= D\int_0^{2\pi}\int_1^3\{369 - 18z^2\}\,dz\,d\theta = 291\pi D$.

11. The area in question is bounded by $\rho = 4a\cos\theta$ and $\rho = 2a$ and by the domain of θ-values determined by the intersection of these two circles. Setting $2a = 4a\cos\theta$, we see that the end-values for θ are $-\pi/3$ and $\pi/3$. Then
$I = (M/A)\int_{-\pi/3}^{\pi/3}\int_{2a}^{4a\cos\theta}\rho^2\rho\,d\rho\,d\theta$. The integration is straightforward except that to handle the $\cos^4\theta$ which results from the first integration one can use the integral formula #73 or replace $\cos^4\theta$ by $(1 - \sin^2\theta)\cos^2\theta$ and then use simple trigonometric identities. The value of I proves to be
$I = (M/A)(a^4/3)(20\pi + 21\sqrt{3})$. We compute the area A which is given by
$\int_{-\pi/3}^{\pi/3}\int_{2a}^{4a\cos\theta}\rho\,d\rho\,d\theta$. This proves to be $(2\pi + 3\sqrt{3})(a^2/3)$. Then I has the text value.

12. We shall use $I = (M/V) \iiint r^2 dV$ and cylindrical coordinates. The distance of any element of volume from the axis of rotation is ρ so that $r^2 = \rho^2$. Now in the upper half of the figure for any fixed ρ and θ, z varies from $\sqrt{b^2 - \rho^2}$ to $\sqrt{a^2-\rho^2}$. ρ varies, for any fixed θ, from c to a and θ from 0 to 2π. Hence since $M/V = d$, and for the entire figure,
$I = 2d \int_0^{2\pi} \int_c^a \int_{\sqrt{b^2-\rho^2}}^{\sqrt{a^2-\rho^2}} \rho^2 \rho \, dz \, d\rho \, d\theta$. The evaluation is straightforward. After one integration we get the terms $\sqrt{a^2-\rho^2}\,\rho^3$ and $\sqrt{b^2-\rho^2}\,\rho^3$. Evaluate each separately. To do the first one, let $\rho = a \sin\phi$. This leads to an integral of $a^5 \sin^3\phi \cos^2\phi$. Replace $\sin^2\phi$ by $1 - \cos^2\phi$ and the rest is direct evaluation. The terms $\sqrt{b^2-\rho^2}\,\rho^3$ gives the same result except that b replaces a. Hence only one somewhat lengthy integration must really be done. The text gives the answer.

13. By using the relations $\sin\theta = y/\rho$, $\cos\theta = x/\rho$, we find perhaps more readily, that the disc is the circle of radius $\sqrt{2}$ shown in Fig. 39. Noting that the area of the circle is 2π, we have
$I = (M/2\pi) \int_{-\pi/4}^{3\pi/4} \int_0^{2(\sin\theta+\cos\theta)} \rho^2 \cdot \rho \, d\rho \, d\theta$
$= (2M/\pi) \int_{-\pi/4}^{3\pi/4} (\sin\theta + \cos\theta)^4 d\theta$. This latter integral can be evaluated by use of the binomial theorem and the integral formulas #72, #73, and #78. The following device may be used instead to shorten the work. We note that $\cos \pi/4 = \sin \pi/4 = 1/\sqrt{2}$. Thus $\sin\theta + \cos\theta = \sqrt{2}[\sin\theta \cos(\pi/4) + \cos\theta \sin(\pi/4)]$ $= \sqrt{2}\sin[\theta + (\pi/4)]$. Hence $I = (8M/\pi) \int_{-\pi/4}^{3\pi/4} \sin^4[\theta + (\pi/4)] d\theta$. The substitution $\psi = \theta + \pi/4$ leads to $I = (8M/\pi) \int_0^{\pi} \sin^4\psi \, d\psi$. Now integral formula #72 leads to $I = 3M$.

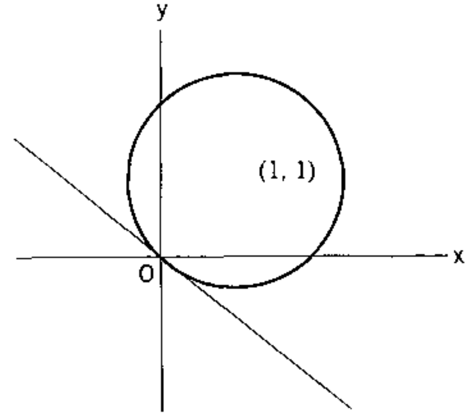


Fig. 39

14. Since the equation of the cone in cylindrical coordinates is $\rho = az/h$, we obtain $I = (M/V) \int_0^{2\pi} \int_0^h \int_0^{az/h} \rho^3 \, d\rho \, dz \, d\theta$. Elementary integrations and use of the formula $V = \frac{1}{3}\pi a^2 h$ leads to $I = \frac{3}{10} Ma^2$.

15. Take the line to be the y-axis. Then $r^2 = x^2 + z^2 = \rho^2\cos^2\theta + z^2$. Using this factor in place of ρ^2 in the integrand of Exercise 14 and using the formula for V, we obtain $I = (M/V) \int_0^{2\pi} \int_0^h \int_0^{az/h} (\rho^3 \cos^2\theta + \rho z^2) d\rho \, dz \, d\theta$
$= (3M/5)\{\frac{1}{4}a^2 + h^2\}$.

16. Here $r^2 = x^2 + y^2 = \rho^2\cos^2\theta \sin^2\phi + \rho^2\sin^2\theta \sin^2\phi = \rho^2\sin^2\phi$. Thus
$I = (M/V)\int_0^{2\pi}\int_0^{\pi/6}\int_0^2 \rho^4\sin^3\phi \, d\rho \, d\phi \, d\theta$ where $V = \int_0^{2\pi}\int_0^{\pi/6}\int_0^2 \rho^2\sin\phi \, d\rho \, d\phi \, d\theta$.
Using integral formula #72 to integrate the $\sin^3\phi$ term in the integral for I or replacing $\sin^2\phi$ by $1 - \cos^2\phi$, we obtain $I = \frac{1}{5}(5 - 2\sqrt{3})M$.

17. Here $r^2 = \rho^2$. Hence $I = (M/V)\int_0^{2\pi}\int_0^4\int_{\rho^2/4}^4 \rho^2 \rho \, dz \, d\rho \, d\theta$. This gives
$I = (M/V)256 \cdot 2\pi/3$. We calculate V from the same integral but without the ρ^2 factor. $V = 32\pi$. Hence $I = 16M/3$.

18. The distance of any element of the volume to the y-axis (the $\theta = \pi/2$ ray) is $\sqrt{z^2 + \rho^2\cos^2\theta}$. Hence $I = (M/V)\int_0^{2\pi}\int_0^4\int_{\rho^2/4}^4 (z^2 + \rho^2\cos^2\phi)\rho \, dz \, d\rho \, d\theta$. The integration is straightforward though lengthy. Then $I = (M/V)512 \cdot 2\pi/3$. The volume V has been computed in Exercise 17 and is 32π. Then $I = 32M/3$.

19. The equation $z = \rho^2$ in cylindrical coordinates is $z = x^2 + y^2$ in rectangular coordinates and so represents a paraboloid.
$I = (M/V)\int_0^{2\pi}\int_0^{\sqrt{2}}\int_{\rho^2}^2 \rho^2 \rho \, dz \, d\rho \, d\theta = 4\pi M/3V$. Here V is the same integral without ρ^2 and proves to be 2π. Hence $I = 2M/3$.

20. (a) Here $r = y = \rho\sin\theta$ and $A = \pi R^2/2$. Thus
$I = (M/A)\int_{-\pi/2}^{\pi/2}\int_0^R \rho^3\sin^2\theta \, d\rho \, d\theta = M\pi R^4/8A = MR^2/4$.

(b) Here $r = x = \rho\cos\theta$ and $A = \pi R^2/2$. Thus
$I = (M/A)\int_{-\pi/2}^{\pi/2}\int_0^R \rho^3\cos^2\theta \, d\rho \, d\theta = MR^2/4$.

(c) $I = (4M/A)\int_0^a\int_0^{b\sqrt{1-x^2/a^2}} y^2 \, dy \, dx$ where $A = 4\int_0^a\int_0^{b\sqrt{1-x^2/a^2}} dy \, dx$. Using integral formulas #31 and #26 for the respective integrals we obtain $I = Mb^2/4$.

(d) Reversing the roles of x and y in (c) we obtain at once $I = Ma^2/4$. Otherwise one may set up the corresponding integrals
$I = (4M/A)\int_0^b\int_0^{a\sqrt{1-y^2/b^2}} x^2 \, dx \, dy$, $A = 4\int_0^b\int_0^{a\sqrt{1-y^2/b^2}} dx \, dy$.

(e) Here $r = y$. Thus $I = (M/A)\int_0^4\int_0^{\sqrt{4x}} y^2 \, dy \, dx$ and $A = \int_0^4\int_0^{\sqrt{4x}} dy \, dx$. We obtain $I = 16M/5$.

(f) Here $r = x$. Thus $I = (M/A) \int_0^4 \int_0^{\sqrt{4x}} x^2\,dy\,dx$. A, found in (e), is $^{32}/_3$. Thus we find $I = 48M/7$.

21. Following the reasoning of the text we have work done $= 32Mx \sin\alpha$ $=$ kinetic energy $= \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$. Thus using $\omega = v/R$ and the given moment of inertia of a cylinder $(I = MR^2/2)$, we have $32Mx \sin\alpha = Mv^2/2$ $+ Mv^2/4 = 3Mv^2/4$ or $v^2 = \frac{4}{3}\,32x \sin\alpha$. At the bottom (see Fig. 23-36) the cylinder has fallen a vertical distance given by $h = x \sin\alpha$. Thus the velocity at the bottom is $v = \sqrt{(4/3)\,32h}$. From the formula for v^2 we obtain, as in the text, $2v(dv/dt) = \frac{4}{3}\,32v \sin\alpha$. Thus $a = dv/dt = \frac{2}{3}\,32 \sin\alpha$. This acceleration is constant all along the path.

22. Let y measure the vertical distance downward from the point of release of the yo-yo. Then work done $= 32My$ and the kinetic energy $= Mv^2/2 + I\omega^2/2$ $= 3Mv^2/4$. As in Exercise 21 $v^2 = \frac{4}{3}\,32y$ and $2v(dv/dt) = \frac{4}{3}\,32v$. Thus $a = dv/dt = {}^{64}/_3$ ft/sec^2.

23. The situation is the same as in Exercise 22 except that now $I = MR^2$. Thus the work done $= 32My$ and the kinetic energy $= Mv^2/2 + I\omega^2/2 = Mv^2/2$ $+ MR^2(v^2/2R^2) = Mv^2$. Hence $32My = Mv^2$ and $v^2 = 32y$. Then $2v\,dv/dt = 32v$ and $dv/dt = 16$ ft/sec^2.

Solutions to Chapter 24

CHAPTER 24, SECTION 2, FIRST SET

1. Use the differential form $\int dy/y^3 = \int x^2 dx$. Hence $y^{-2}/-2 = x^3/3+C$, which can of course be rewritten in many simple forms.
2. Using the differential form gives $\int y^3 dy = \int x^2 dx$ or $y^4/4 = x^3/3+C$.
3. Write $dy/dx = e^x e^{-y}$ and now use the differential form $\int e^y dy = \int e^x dx$ so that $e^y = e^x+C$.
4. The problem is already in differential form. To integrate use #26 in the table of integrals for the right side and #40 in the table of integrals for the left side.
5. The differential form is $y\ dy = (\log x/x)dx$. Since the derivative of $\log x$ is $1/x$ we have, by integrating, $y^2/2 = (\log x)^2/2+C$.
6. The differential form is $y\ dy/y^2+3) = x\ dx/(1+x^2)$. Integration gives $\log(y^2+3)/2 = \log(x^2+1)/2+C$ or $y^2+3 = C(x^2+1)$.
7. Separation of variables leads to $\sec^2 x\ dx = -\csc^2 y\ dy$. Both sides are immediately integrable and one obtains $\tan x = +\cot y+C$. Since when $x = \pi/4$, $y = \pi/4$, $C = 0$.
8. Written as $xdx/\sqrt{1+x^2} = ydy/\sqrt{1+y^2}$ both sides are immediately integrable by letting $u = 1+x^2$ on the left and $u = 1+y^2$ on the right. Then $\sqrt{1+x^2} = \sqrt{1+y^2}+C$.
9. Writing y' as dy/dx suggests separation of variables. We get $dy/\ y = (8x+3)dx$. Hence $\log y = 4x+3x+C$ or $y =$

   $$e^{4x^2+3x+C} = e^C e^{4x^2+3x} = De^{4x^2+3x}$$
10. Letting $v = x+y$ and treating x still as the independent variable gives $dv/dx = 1+dy/dx$. Hence the original differential equation becomes $dv/dx - 1 = v^2$ or $dv/dx = 1+v^2$. Now use separation of variables and obtain $\arctan v = x+C$ or $\arctan(x+y) = x+C$.

CHAPTER 24, SECTION 2, SECOND SET

1. (a) With (12) as the standard form $P(x) = 1/x$ and $Q(x) = 1$. Hence the integrating factor $e^{\int P(x)dx}$ becomes $e^{\log x} = x$ and we multiply the original equation by x and obtain $x(dy/dx)+y = x$ or $d(xy)/dx = x$. Integration gives $xy = x^2/2+C$.

   (b) Following the method as just presented in exercise (a), the integrating factor is $e^{\int 3dx/x} = e^{3\log x} = e^{\log x^3} = x^3$. Hence the original differential equation becomes $x^3y'+3x^2y = x^4$ or $d(x^3y)/dx = x^4$. Integration yields $x^3y = x^5/5+C$.

   (c) As in the method of exercise (a), the integrating factor is $e^{\int 2dt} = e^{2t}$. Hence $e^{2t}dr/dt+2re^{2t} = 10$ or $d(e^{2t}r)/dt = 10$. Integrating gives $re^{2t} = 10t+C$. Since $r = 0$ when $t = 0$, $C = 0$.

(d) Here the integrating factor is $e^{\int 10dx/(2x+5)} = e^{5 \log(2x+5)}$ $= (2x+5)^5$. Hence multiplying both sides of the original differential equation by $(2x+5)^5$ gives $d[(2x+5)^5 y]dx = 10(2x+5)^5$. Integration yields $(2x+5)^5 y = 5(2x+5)^6/6+C$.

(e) We can write at once $d(xy)dx = \sin x$. Then $xy = -\cos x+C$.

2. The given amounts to $dy/dx - 3y = 2x$. The integrating factor is $e^{\int -3dx} = e^{-3x}$. Multiplying through by this we obtain $d(ye^{-3x})/dx = 2xe^{-3x}$. To integrate we must use integration by parts on the right side or a table of integrals. The result is $ye^{-3x} = (-2xe^{-3x}/9)-(2e^{-3x}/9)+C$.

CHAPTER 24, SECTION 2, THIRD SET

1. Since the differential equation is $dv/dt+kv = 32$, the integrating factor is $e^{\int kdt}$ or e^{kt}. Then by multiplying through by e^{kt}, the original equation becomes $d(ve^{kt})/dt = 32e^{kt}$. Integration yields $ve^{kt} = (32/k)e^{kt}+C$.
2. The equation to be solved is $di/dt+5i = 10 \cos 5t$. The integrating factor is $e^{\int 5dt}$ or e^{5t}. Multiplying through by e^{5t} and integrating yield $e^{5t}i = 10\int e^{5t}\cos 5t$. The right side is integrated by integration by parts or by using a table. This gives $e^{5t}i = e^{5t}(\cos 5t + \sin 5t)+C$. In practice $i = 0$ when $t = 0$ so that $C = -1$.
3. The differential equation is $2di/dt+40i = 20$. The method of linear equations used throughout this section yields $i = (1/2)(1-e^{-20t})$.
4. The first part of the problem calls for solving $.1di/dt+10i = 10$ or $di/dt+100i = 100$. The method of linear equations yields $i = 1+Ce^{-100t}$. However, $i = 0$ when $t = 0$ so that $i = 1-e^{-100t}$. This certainly holds for $0 < t < 5$. The second part of the problem calls for solving $.1di/dt+10i = 0$ or $di/dt+100i = 0$. Here straightforward calculus using $dt/di = -1/100i$ yields $i = De^{-100t}$ where D is the constant of integration. However, from the first part of the problem when $t = 5$, $i = 1-e^{-500}$. Hence to determine D we set $1-e^{-500} = De^{-500}$ or $D = e^{500}-1$. Hence $i = (e^{500}-1)e^{-100t}$ for $t > 5$.

CHAPTER 24, SECTION 3

1. (a) By substituting e^{mx} in $y''-y = 0$ we see that the characteristic equation is $m^2-1 = 0$. Then $m = \pm 1$ and by (20) the general solution is the answer in the text.

(b) By substituting e^{mx} in the given differential equation we obtain $m^2-3m-10 = 0$. The roots of this quadratic equation are 5 and -2. By (20) the general solution is the answer in the text.

(c) As in (a) and (b) we obtain the characteristic equation and the roots $(-1 \pm \sqrt{5})/2$. Since the roots are real and distinct we still use (20) and the general solution is $y = Ae^{(-1+\sqrt{5})x/2}+Be^{(-1-\sqrt{5})x/2}$.

(d) As in (a) and (b) we obtain the characteristic equation $m^2+m+1 = 0$. The roots are $(-1\pm\sqrt{-3})/2$. Since the roots are complex we can use (22) and write the general solution as $y = Ae^{(-1+\sqrt{-3})x/2}+Be^{(-1-\sqrt{-3})x/2}$. However, the text shows that we can also use (35) so that $y = Ae^{-x/2}\cos\sqrt{3}\ x/2+Be^{-x/2}\sin\sqrt{3}\ x/2$.

(e) As in (a) and (b) the characteristic equation turns out to be $m^2+4m+4 = 0$ and the roots are $-2,-2$. Since the roots are equal we use (21) and the general solution is $y = e^{-2x}(A+Bx)$.

(f) This problem like (e) leads to the equal roots $-3,-3$. Hence the general solution is $y = e^{-3x}(A+Bx)$.

2. This exercise is to prove that (21) is correct. We know that one solution is e^{mx}. We try $y = u(x)e^{mx}$ as the possible form of the general solution. By substituting in the given differential equation and rearranging terms we get $u''(x)+u'(x)(2m+2\alpha)+u(x)(m^2+2\alpha m+\beta) = 0$. However, m satisfies the characteristic equation so that the coefficient of $u(x) = 0$. Moreover, since m is a double root, the discriminant of the quadratic formula must be 0 and so $m = -\alpha$. Hence the coefficient of $u'(x)$ is 0. Then $u''(x) = 0$ and $u = Ax+B$. Then the general solution is $y = (A+Bx)e^{mx}$ where $m = -\alpha$.

3. (a) There is no radial force acting on the bead. Hence the radial acceleration is 0; that is, $\ddot{\rho}-\rho\dot{\theta}^2 = 0$. Then since $\dot{\theta} = \omega$ is a constant, $\ddot{\rho}-\rho\omega^2 = 0$. The characteristic equation is $m^2-\omega^2 = 0$ or $m = \pm\omega$. Then the general solution is $\rho = Ae^{\omega t}+Be^{-\omega t}$. Since $\dot{\theta} = \omega$, $\theta = \omega t+C$ and because $\theta = t$ when $t = 0$ we can write the polar coordinate equation $\rho = Ae^{\theta}+Be^{-\theta}$.

(b) Since $\rho = 0$ when $\theta = 0$, $B = -A$. Then $\rho = Ae^{\theta}-Ae^{-\theta}$. For $\theta \geq 0$ $e^{\theta} > e^{-\theta}$ and the inequality becomes larger as θ increases. Hence the bead starts at the origin and moves out continuously. One can find ρ' and see that it is never zero. Hence this is a check that the bead does not reach a maximum position.

Physicists would probably prefer to solve this problem differently. The wire is a rotating system and the bead experiences a centrifugal force of $m\rho\omega^2$ and a Corolis force which is offset by the pressure of the wire on the bead. Then Newton's second law says that $m\ddot{\rho} = m\rho\omega^2$. The rest of the work is the same as above.

CHAPTER 24, SECTION 4

1. To find the particular integral substitute ae^{2x} in the differential equation. Then $13ae^{2x} = 4e^{2x}$ so that $a = 4/13$. Hence the answer in the text.
2. Since the roots of the characteristic equation are equal and both are -2 the complementary function is $(A+Bx)e^{-2x}$. To find the particular integral assume $y_p = a \sin 3x + b \cos 3x$. Substitution in the equation leads to
$$(-5a-12b)\sin 3x + (12a-5b)\cos 3x = 6 \sin 3x.$$
Then
$$-5a-12b = 6$$
$$12a-5b = 0$$
Hence $a = -\frac{30}{169}$ and $b = -\frac{72}{169}$ so that the particular solution is
$$y_p = -\frac{30}{169} \sin 3x - \frac{72}{169} \cos 3x .$$
3. The complementary function is $y_c = e^{-2x}(A \cos\sqrt{5}\,x + B \sin\sqrt{5}\,x)$. Substituting ax^2+bx+c in the differential equation leads to
$$ax^2+(8a+b)x+2a+4b+c = x^2+5x.$$
Equating coefficients of like powers of x shows that
$a = 1$, $8a+b = 5$, so that $b = -3$, and
$2a+4b+c = 0$ so that $c = 10$. Hence $y_p = x^2-3x+10$.
4. Because the roots of the characteristic equation are both -1, $y_c = (A+Bx)e^{-x}$. Following the example of the text leads to $y_p = -\frac{12}{25} \sin 2x - \frac{16}{25} \cos 2x$.
5. The complementary function $y_c = Ae^{-2x}+Be^{-x}$. Assuming $y_p = ae^{-2x}$ will not do because no matter what a is, ae^{-2x} is part of y_c and will not yield $3e^{-2x}$. If one tries $y_p = xae^{-2x}$ he finds after substituting in the differential equation that $-ae^{-2x} = 3e^{-2x}$ so that $a = -3$. Hence $y_p = -3xe^{-2x}$.
6 (a) $y''+9y = \sin 3x$.
The characteristic equation is $m^2+9 = 0$ and the roots are $\pm 3i$. Hence $y_c = A \sin 3x + B \cos 3x$. Now if we try $y_p = a \sin 3x$ for the particular integral it will not produce $\sin 3x$ because we see that y_c already contains a $\sin 3x$ term and this satisfies the homogeneous equation. Hence we try $x(a \sin 3x + b \cos 3x)$. By substituting in the original d.e. we find that $a = 0$ and $b = -1/6$. Then the full solution is
$$y = y_c+y_p = A \sin 3x + B \cos 3x - \frac{1}{6}x \cos 3x.$$
(b) The complementary function will remain a finite amplitude sinusoidal or cosinusoidal function whatever A and B might be.

But the particular integral, because of the factor x will give rise to larger and larger oscillations and for large x will dominate the behavior of y. If the differential equation represented the motion of a mass on a spring subject to the external force sin 3x, the oscillations will become so large that Hooke's law will no longer apply and in fact the spring will snap. When the frequency of the external force, $3/2\pi$ in this case, equals the natural frequency, then since there is no damping, that is no y' term, the oscillations become infinite. This is called the resonance case. If there is damping the transient dies out and the oscillations due to an external periodic force can become large for the proper range of frequencies but the forced oscillations will not become infinite.

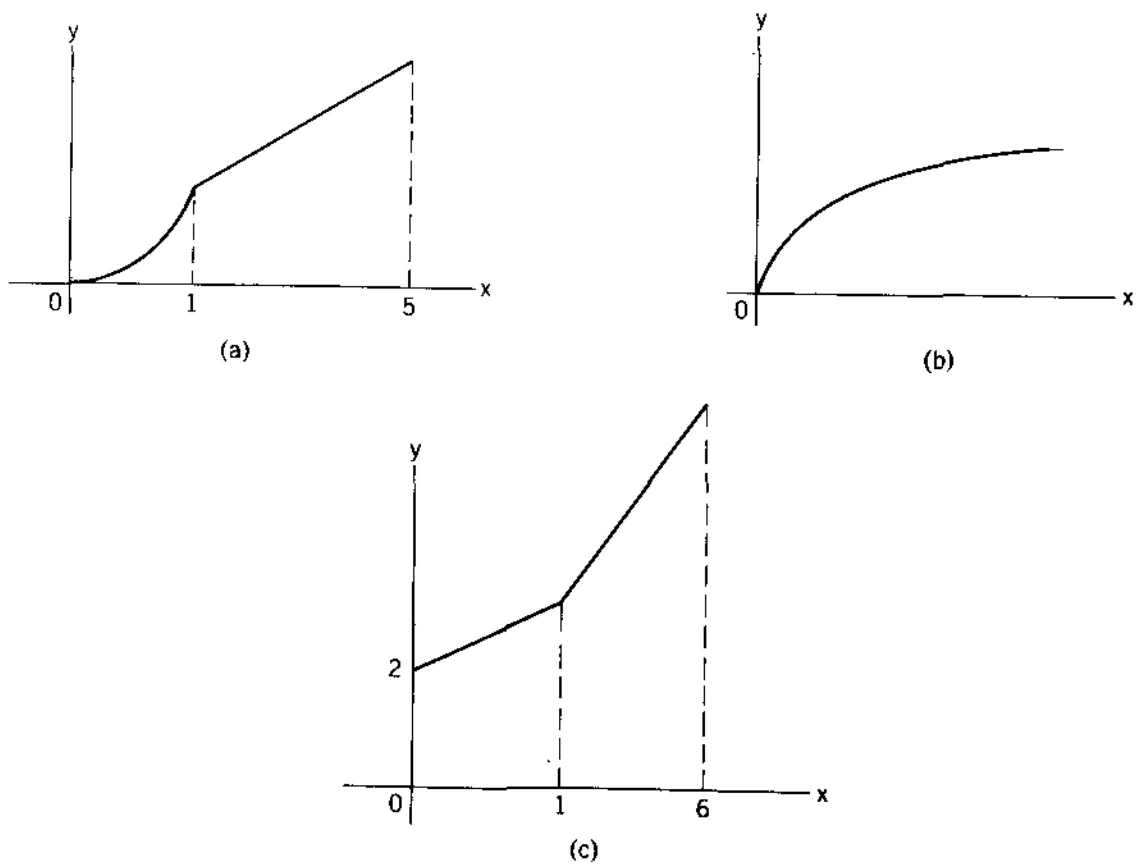
Solutions to chapter 25

CHAPTER 25, SECTION 2

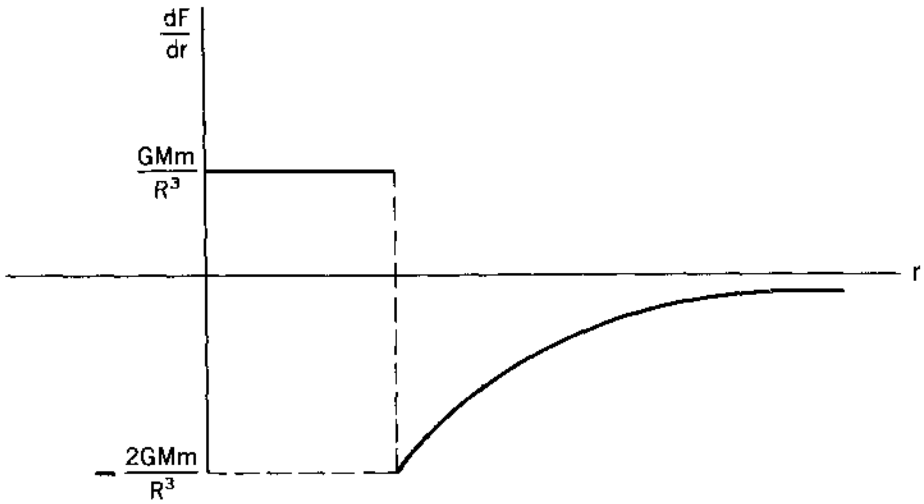
1. For each x in a given domain, y has a unique value.
2. The domain of a function is the set of values for which the function is defined. The range of a function is the set of values taken on by the function.
3. (a) $0 \leq x < \infty$; $y = 10, 20, 30, \ldots$.

   (b) r can be any value > 0; F can have any positive value from 0 to GM/R^2.

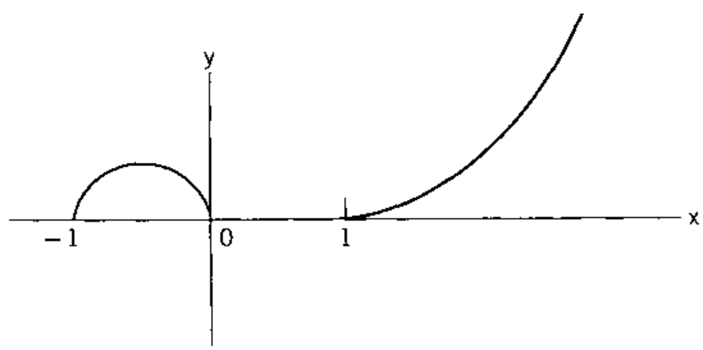
4.



5. The slope is given by $y = dF/dr$. Thus for $0 < r < R$ the slope is $y = GMm/R^3$, which is a constant, whereas for $r > R$ the slope is $y = -2GMm/r^3$.

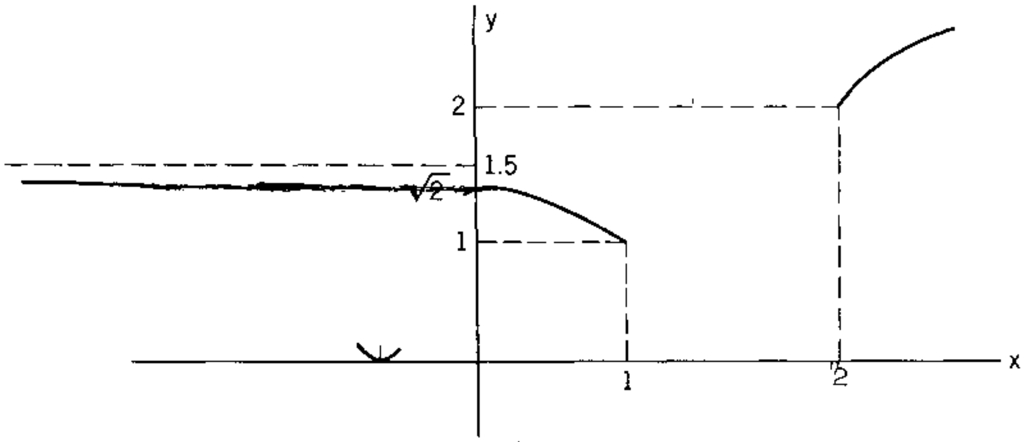


6. During the first 6 seconds we have $\ddot{y} = -32$ and $\dot{y} = 400$ at $t = 0$. Thus $v = -32t + 400$ $(0 \le t \le 6)$. At $t = 6$, $v = 208$. For $t > 6$ we have $\ddot{y} = -32$, and $\dot{y} = -32t + C$. When $t = 6$, $\dot{y} = 308$. Hence $C = 500$. Then $\dot{y} = -32t + 500$ for $6 \le t \le 15$.
7. The domain consists of the x-values such $x^3 - x > 0$. Since $x^3 - x = x(x^2 - 1)$, this inequality holds for $-1 \le x \le 0$ and $x \ge 1$. Thus the domain consists of these two disjoint intervals. For $1 \le x < \infty$, y takes all positive values and so the range is $0 \le y < \infty$.



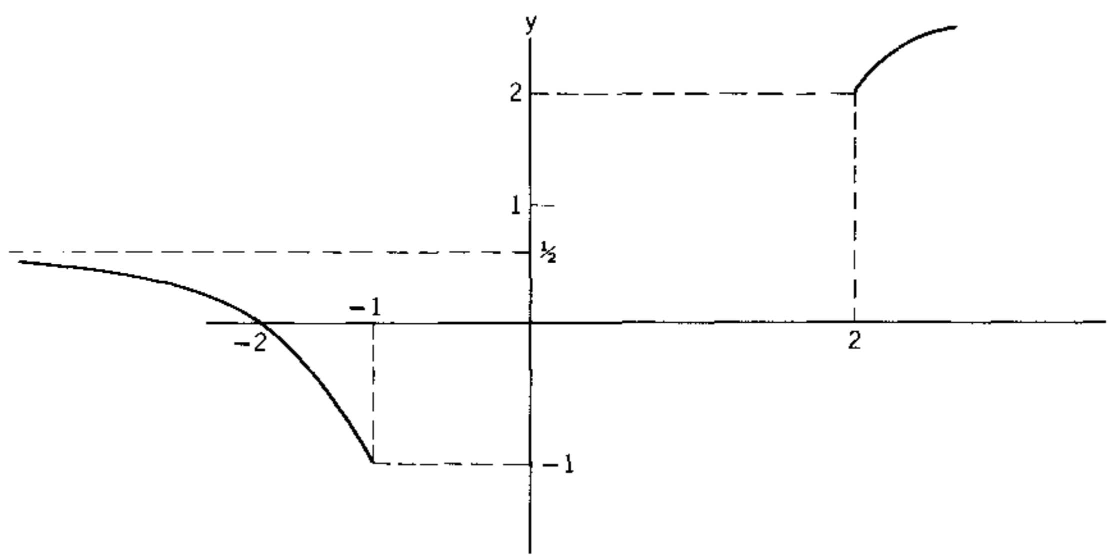
8. To graph the functions accurately would call for using all information including the value of the derivative. What is significant for our purpose is the domain and range.

(a)



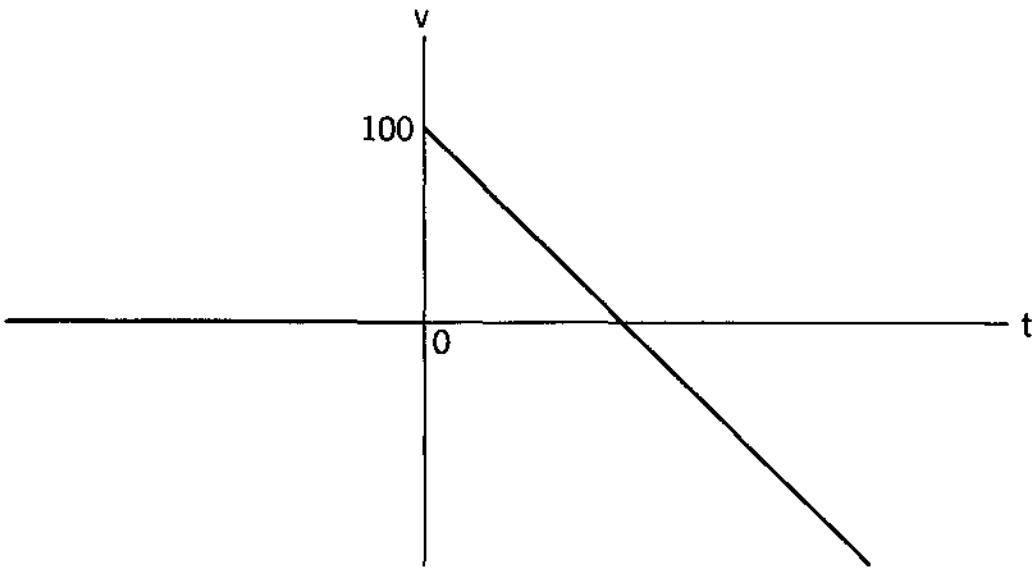
(a)

(b)



(b)

9. For $t < 0$, $v = \dot{y} = 0$. For $t > 0$ we have $\ddot{y} = -32$, $\dot{y} = 100$ at $t = 0$. Thus for $t > 0$, we obtain $v = -32t + 100$.



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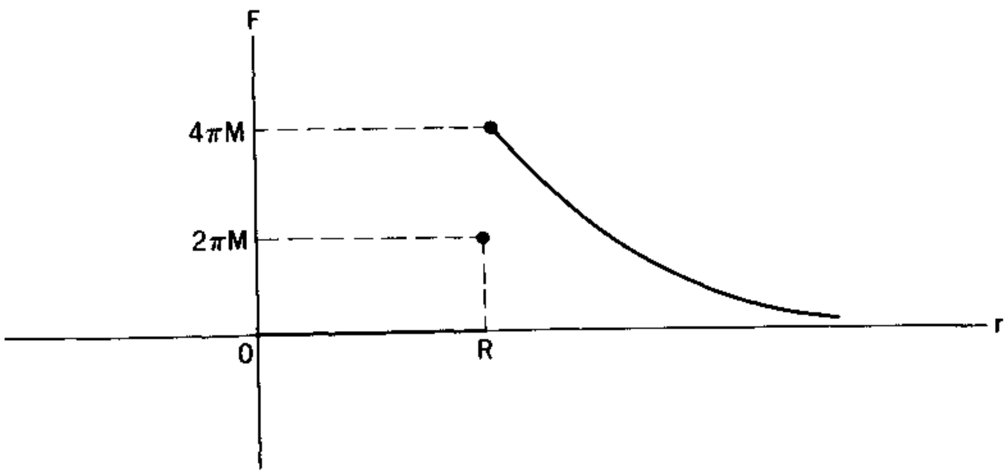

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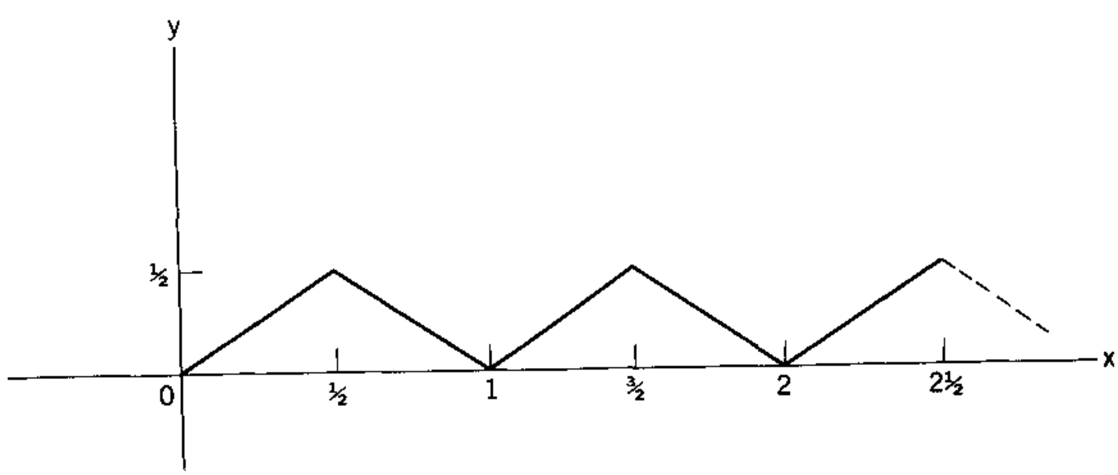

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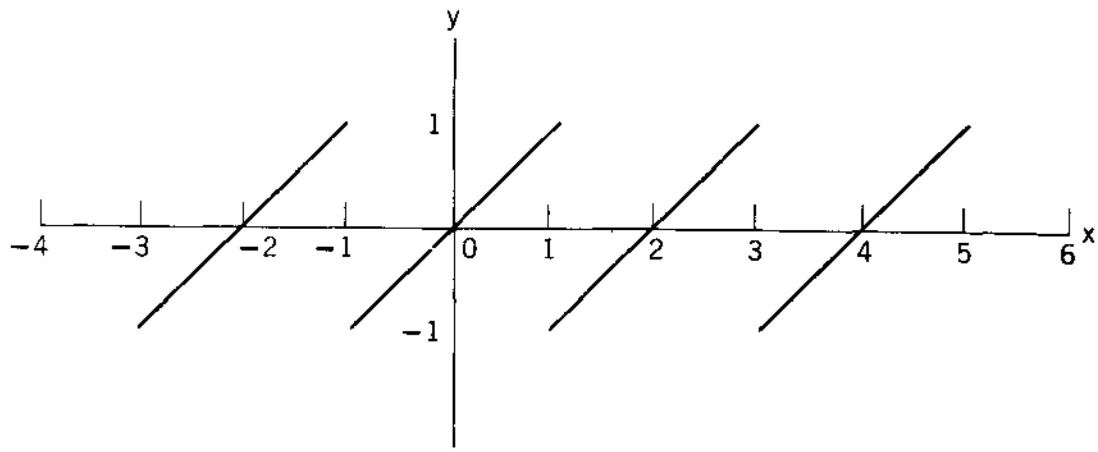

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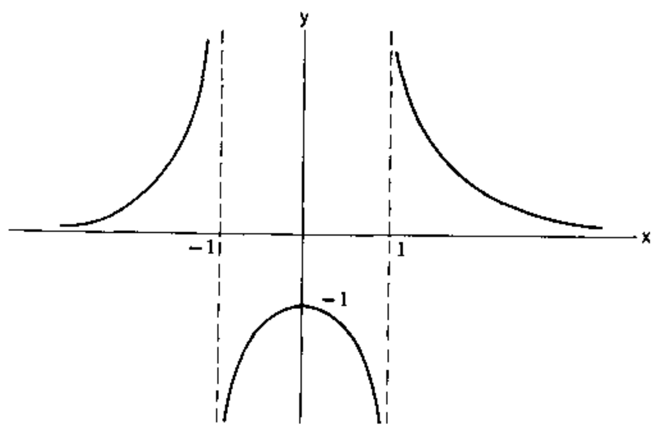

CHAPTER 25, SECTION 3

1. (a) y approaches 0 as x approaches 0.
   (b) y approaches 0 as x approaches 0.
   (c) There is no limit as x approaches 0. The function oscillates more and more rapidly between $y = -1$ and $y = +1$ as x approaches 0.
   (d) Note that the graph of the function is contained within the lines $y = \pm x$. y approaches 0 as x approaches 0.
   (e) $\lim_{x \to 0} \sin 2x/x = \lim_{x \to 0} 2 \sin 2x/2x = 2$; thus y approaches 2 as x approaches 0.
   (f) Graphically or by the kind of argument in (e) we see that y approaches $\frac{1}{2}$ as x approaches 0.
   (g) y approaches 1 as x approaches 1.
   (h) No limit. As x approaches 0 through negative values y approaches 0, but as x approaches 0 through positive values, y becomes infinite.

(i) y approaches 0 as x approaches 0.
(j) No limit. As x approaches 0 through negative values, y becomes infinite but as x approaches 0 through positive values y approaches 0.
(k) y approaches 0 as x approaches 0.
(l) No limit. As x approaches 0 through negative values y approaches -1, but as x approaches 0 through positive values y approaches $+1$.
(m) y becomes infinite as x approaches 1.
(n) No limit.
(o) No limit. As x approaches 0 through negative values y approaches $-\pi/2$, but as x approaches 0 through positive values, y approaches $\pi/2$.
(p) No limit. See text answer.
(q) y approaches 0 as x approaches 0.

2. (a) The condition $|4x - 8| < \epsilon$, is equivalent to $4|x - 2| < \epsilon$ or $|x - 2| < \epsilon/4$. Thus choosing $\delta = \epsilon/4$ we have for all x such that $0 < |x - 2| < \delta$ it is true that $|4x - 8| < \epsilon$.
   (b) Since $|3x - 9| < \epsilon$ is equivalent to $3|x - 3| < \epsilon$ or $|x - 3| < \epsilon/3$, we choose δ to be $\epsilon/3$. Then when $0 < |x - 3| < \delta$, $|3x - 9| < \epsilon$.
   (c) $|(4x + 7) - 15| < \epsilon$ is equivalent to $|4x - 8| < \epsilon$ or $|x - 2| < \epsilon/4$. Choose δ to be $\epsilon/4$. Then when $0 < |x - 2| < \delta$, $|4x + 7 - 15| < \epsilon$.
   (d) $|4 - 4| < \epsilon$ is always valid. Thus it holds for $0 < |x - 2| < \delta$ where δ is completely arbitrary. Any δ will do.
   (e) We can keep $|x - 2| < \epsilon$ if we choose δ to be ϵ, for then, when $0 < |x - 2| < \delta$ it will also be true that $|x - 2| < \epsilon$.
3. The x-values for which $0 < |x - a| < \delta/2$ are contained within the x-values for which $0 < |x - a| < \delta$. Since for all the latter x's, $|y - b| < \epsilon$, it is certainly true for the smaller set of x-values.
4. Yes. The argument is the same as in Exercise 3, except that here the number less than δ replaces the $\delta/2$ in Exercise 3.
5. The idea behind the proof is that if f(x) comes arbitrarily close to b as x approaches a, f(x) cannot come arbitrarily close to another distance number c which is $c - b$ away from b. To make the proof we suppose f(x) does have b and c as a limit. Then given any ϵ, there is a δ_1 such that $|f(x) - b| < \epsilon$ for $0 < |x - a| < \delta_1$ and for the same ϵ, there exists a δ_2, say, such that $|f(x) - c| < \epsilon$ for $0 < |x - a| < \delta_2$. Now $|b - c| = |b - f(x) + f(x) - c| < |b - f(x)| + |f(x) - c|$. For $|x - a| < \delta$, where δ is the smaller of δ_1 and δ_2, certainly $|b - f(x)| < \epsilon$ and $|f(x) - c| < \epsilon$. Then $|b - c| < 2\epsilon$. Now this is true for any ϵ. But it cannot be true for $\epsilon < (b - c)/2$. Hence a contradiction.

6. (a) As x approaches 0, the numerator approaches 1 and the denominator approach 0.
   (b) As x approaches 1, the numerator approaches 1 and the denominator approaches 1. Hence the limit is 1.
   (c) Same as (a).
   (d) To find the limit we can divide numerator and denominator by x because the value at $x = 0$ does not matter as far as the limit is concerned. Then $(x - 1)/1$ approaches -1 as x approaches 0.
   (e) As in (d), we can divide numerator and denominator by $x + 1$. This gives $x^2 - x + 1$. As x approaches 1, this quantity approaches 1.
   (f) As in (d), we may divide numerator and denominator by $x - 2$. This gives $1/(x + 1)$. This quantity approaches $\frac{1}{3}$ as x approaches 2.
   (g) If we use the suggestion we obtain $2x/x(\sqrt{1+x} + \sqrt{1-x})$. As in (d), we may cancel the x's. Then the quantity approaches 1 as x approaches 0.
   (h) If we divide numerator and denominator by x we have $1/[(|x|/x) + x]$. If x approaches 0, through negative values, $|x|/x$ approaches -1 and the entire fraction approaches -1. If x approaches 0 through positive values the entire fraction approaches 1. Hence there is no limit.
7. It is intuitively clear that $4x^2$ approaches 4. Since, by Exercise 5, there cannot be two limits, 5 is not the limit.

CHAPTER 25, SECTION 4

1. (a) We may take $\lim(x + 4)$ and divide it by $\lim x$. But $\lim(x + 4) = \lim x + \lim 4$. Hence the limit of the entire quantity is 3.
   (b) $\text{Lim}(5x) = \lim 5 \cdot \lim x = 10$.
   (c) $\text{Lim}\, x^2 = \lim x \cdot \lim x = 9$.
   (d) We may take $\lim(x^2 - 5x)$ and divide it by $\lim(x^2 + 4x + 6)$. Now $\lim(x^2 - 5x) = \lim x^2 - \lim 5x = \lim x \cdot \lim x - \lim 5 \cdot \lim x = -6$. Breaking up the denominator in a similar way we find $\lim(x^2 + 4x + 6) = 18$. Hence the limit of the fraction is $-\frac{1}{3}$.
   (e) The limit of the sum is the sum of the limits. But $\lim 3x_0^2 = 3x_0^2$; $\lim 3x_0\,\Delta x = \lim 3x_0 \lim \Delta x = 0$ and $\lim(\Delta x)^2 = \lim \Delta x \lim \Delta x = 0$. Hence the text result.
2. $\text{Lim}\, x$ is of course 0. The function $\sin(1/x)$ oscillates rapidly as x approaches 0 and does not approach a limit. (see Section 3, Exercise 1(c)). However $x\sin(1/x)$ does approach 0 because the values of $\sin(1/x)$ are always between -1 and 1 and when multiplied by x, the product approaches 0. Theorem 3 does not apply because the separate limits do not both exist in the present case.

3. $[(-b \pm \sqrt{b^2-4ac})/2a](-b \mp \sqrt{b^2-4ac})/(-b \mp \sqrt{b^2-4ac}) = 2c/(-b \mp \sqrt{b^2-4ac})$. Now if a approaches 0, then when the − radical holds, the limit is $-c/b$. When the + radical holds the limit is infinite.
4. We use theorem 3 to assert that the limit of the product is m/n.
5. No. As x approaches 0 through positive values we must use x^2+1 and its limit is 1. As x approaches 0 through negative values, we must use $-(x^2+1)$ and its limit is -1.
6. (a) As x approaches 1 we are concerned only with values of x near 1 (but not 1 itself). For these values of x, $g(x) = 1/x$ and $1/x$ approaches 1.
   (b) The argument is the same at $x = 2$ as at $x = 1$. The limit of $1/x$ as x approaches 2 is $\frac{1}{2}$.
   (c) Again the exceptional values of x, that is, 0, 1, 2, ... do not matter as x approaches 1. Then $g(x)/x = 1/x^2$ and the limit is 1.
   (d) Same argument as (c). Hence the limit is $\frac{1}{4}$.

CHAPTER 25, SECTION 5, FIRST SET

1. Yes, the definition of continuity requires a limit for f(x).
2. Yes, because $\lim_{x\to 2} x^2 = f(2)$.
3. Yes, because the product of two continuous functions is continuous.
4. No. $(\sin x)/x$ is not defined at $x = 0$. Hence there is no $f(0)$.
5. No, because the function is not defined at $x = 0$.
6. (a) Yes. Consider $f(x) = 0$ for $x \le 0$ and $f(x) = \frac{3}{4}$ for $x > 0$ and let $a = 0$. Then for $\epsilon = \frac{1}{2}$ it is not true that $|f(x) - f(0)| < \frac{1}{2}$ when x is positive and so no δ exists. But if we let $\epsilon = 1$, then $|f(x) - f(0)| < 1$ for all x and any δ will do.
   (b) No. Decreasing ϵ to $\frac{1}{4}$ makes matter worse, so to speak. If there is no δ such that when $|x - a| < \delta$, then $|f(x) - f(a)| < \frac{1}{2}$, there certainly won't be one for which $|f(x) - f(a)| < \frac{1}{4}$.
7. No, because the definition of continuity requires that there exist a δ such that $|f(x) - f(a)| < \epsilon$ for every (positive) ϵ. The function f(x) might have a jump at $x = a$ of magnitude 0.001 and so meet the $\epsilon - \delta$ condition for all ϵ down to 0.001 but not for smaller ϵ.
8. Yes. Suppose some one picks an ϵ. Then we must show there exists a δ such that when $|x - a| < \delta$, $|f(x) - f(a)| < \epsilon$. We can always find a fraction $1/2^n$ such that $1/2^n < \epsilon$. For this $1/2^n$ we know there is a δ for which $|f(x) - f(a)| < 1/2^n$. Then surely $|f(x) - f(a)| < \epsilon$ with the same δ.
9. (a) Note first that $f(3) = 6$. Then we consider $f(x) - f(3)$. Now $|2x - 6| < \epsilon$ is equivalent to $|x - 3| < \epsilon/2$. Let $\delta = \epsilon/2$. Then for $|x - 3| < \delta$, $|2x - 6| < \epsilon$.

(b) We consider $f(x) - f(2)$. $|4x + 6 - 14| = |4x - 8| = 4|x - 2|$. Then $4|x - 2| < \epsilon$ when $|x - 2| < \epsilon/4$. Let $\delta = \epsilon/4$.

(c) $f(x) - f(1) = 3x - 7 + 4 = 3x - 3$. Now $|3x - 3| < \epsilon$ when $|x - 3| < \epsilon/3$. Let $\delta = \epsilon/3$.

(d) $f(x) - f(2) = 5x + 7 - 17$. Now $|5x + 7 - 17| = |5x - 10| = 5|x - 2|$. Then $5|x - 2| < \epsilon$ when $|x - 2| < \epsilon/5$. Let $\delta = \epsilon/5$.

10. Yes. Note that $f(0) = 0$. As x approaches 0 through positive values x^2 approaches 0 and as x approaches 0 through negative values $-x^2$ approaches 0.

11. (a) No value.

(b) Yes. We know (Section 4, Exercise 2) that $\lim\limits_{x\to 0} x\sin(1/x) = 0$. If we define $f(0)$ to be 0 we satisfy the definition of continuity.

12. Review Exercise 6 of Section 4

(a) Yes. At $x = 1$, $g(x)$ is defined to be 1. As x approaches 1, $g(x) = 1/x$ and $1/x$ approaches 1.

(b) No. At $x = 2$, $g(x)$ is defined to be 1. But as x approaches 2, $g(x) = 1/x$ approaches $\frac{1}{2}$. Then $\lim\limits_{x\to 2} g(x) \neq g(2)$.

(c) Yes. At $x = 1$, $g(x)$ is defined to be 1 and $g(x)/x = 1$. As x approaches 1, $g(x) = 1/x^2$ approaches 1.

(d) No. At $x = 2$, $g(x)$ is defined to be 1 and $g(x)/x = \frac{1}{2}$. But as x approaches 2, $g(x)/x = 1/x^2$ approaches $\frac{1}{4}$. Then $\lim\limits_{x\to 2} (g(x)/x) \neq g(2)/2$.

CHAPTER 25, SECTION 5, SECOND SET

1. We give the answers not in the text. The answers are obtainable readily from a good sketch or by examining the functions.

   (b) Yes; no. (d) Yes; no. (f) No; no. (h) No; no. (i) No; no.

2. Yes. Insofar as finding the limit at $x = 0$, the value of the function at $x = 0$ is irrelevant. But for $x \neq 0$, we may divide numerator and denominator by x.

3. It is all right to divide by $x - 3$ insofar as finding the limit as x approaches 3 is concerned. However to find $\lim\limits_{x\to 3} x + 3$ by substituting 3 for x presupposes the continuity of the function $x + 3$ at $x = 3$. If one has proved this continuity or knows it on some ground then the substitution is correct. One must add that $x + 3$ is continuous at $x = 3$ to make the reasoning complete.

CHAPTER 25, SECTION 6

1. (a) 1, 4, 9, 16, 25.
   (c) $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}$.
   (d) $0, \frac{1}{5}, \frac{2}{7}, \frac{1}{3}, \frac{4}{11}$.
2. We have for $\epsilon = \frac{1}{5}$ that $|S - s_n| < \epsilon$ for $n > 5$. Then surely for $n > 3$, $|S - s_n| < \epsilon$. What holds for $\epsilon = \frac{1}{10}$ need not hold for $\epsilon = \frac{1}{20}$ because the latter ϵ is smaller. Since S is the limit, for $\epsilon = \frac{1}{20}$ there is some N such that for $n > N$, $|S - s_n| < \epsilon$ but N need not be 5.
3. (a) We may take $N = 10$ (or larger) because for $n > 10$, $|1/n - 0| < \epsilon$.
   (b) We may take $N = 4$ (or larger) because for $n > 4$, $|(1/2^n - 0)| < \epsilon$.
   (c) This the same as (b) because the minus signs do not affect the absolute value.
4. The condition that there exists an N such that for $n > N$, $|S - s_n| < \epsilon$ can be met (if at all) by letting N be 150 or 200 or some other value, depending upon what ϵ is. If the first 100 terms do not meet the ϵ condition it does not matter. The essential point is that given any ϵ there be some N (definite for that ϵ but no matter how large) such that $|S - s_n| < \epsilon$.
5 (a) Here $|S - s_n| = |2 - (2n-2)/n| = |2/n|$. Given ϵ, $|2/n| = 2/n < \epsilon$ is equivalent to $n > 2/\epsilon$. Hence choose N to be the first integer greater than $2/\epsilon$. Then for $n > N$, surely $n > 2/\epsilon$ and $|2/n| < \epsilon$.
   (b) Here $|S - s_n| = |2 - (2^{n+1} - 1)/2^n| = |1/2^n|$. Given ϵ, $1/2^n < \epsilon$ is equivalent to $n > \log(1/\epsilon)/\log 2$. Hence choose N to be the first integer greater than $\log(1/\epsilon)/\log 2$. Then for $n > N$ we surely have $n > \log(1/\epsilon)/\log 2$ and therefore $|1/2^n| < \epsilon$.
   (c) Here $|S - s_n| = |0 - (1/2^n)|$. This is what (b) calls for.
6. Letting S be $\frac{3}{2}$ we have for $|S - s_n|$, $|\frac{3}{2} - (2^n - 1)/2^n|$ or $|\frac{3}{2} - 1 + 1/2^n|$ or $|\frac{1}{2} + 1/2^n|$. If we choose $\epsilon = \frac{1}{2}$, say, then $|\frac{1}{2} + 1/2^n|$ will not be less than ϵ for any n. Hence for at least one ϵ there is no N such that for $n > N, |S - s_n| < \epsilon$.
7. For the s_{2m} terms, $|0 - 1/m| < \epsilon$ for $m > 1/\epsilon$. For the s_{2m-1} terms, $|0 - (1/2m)| < \epsilon$ for $m > 1/2\epsilon$. If we choose N to be the first integer greater than $1/\epsilon$, then N is also greater than $1/2\epsilon$. Here for these n, $|0 - 1/m| < \epsilon$ and $|0 - 1/2m| < \epsilon$.
8. $|1 - 1/n| > \frac{1}{2}$ for all $n > 2$. Hence if we choose $\epsilon = \frac{1}{2}$ there is no N such that for $n > N$, $|1 - 1/n| < \frac{1}{2}$. Thus for at least one ϵ there is no N and the definition is not satisfied.

CHAPTER 25, SECTION 7

1. (a) Write $n/(n^2 + 2)$ as $(1/n)/(1 + 2/n^2)$. To use theorem 4 we need the limit of the denominator. This is evaluated by using theorem 1. Hence the answer is 0.
   (b) Write s_n as $(1 + 1/n)/(1 + 2/n)$. Now apply theorem 4 and then apply the sum theorem to the numerator and denominator. The answer is 1.
   (c) Write s_n as $(1 - 3/n + 1/n^2)/(2 - 4/n + 1/n^2)$. Now use theorem 4 and then apply theorem 1 and 2 to the numerator and denominator.
   (d) Write s_n as $(2 + 1/n)/(3 + 1/n)$. Now use theorem 4 and then theorem 1. The answer is $2/3$.
2. If $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b$ then for arbitrary ϵ, there exists an N_1 such that $n > N_1$ implies $|a - a_n| < \epsilon/2$ and there exists an N_2 such that for $n > N_2$, $|b - b_n| < \epsilon/2$. Thus for N equal to the larger of N_1 and N_2 we have that for $n > N$, $|(a + b) - (a_n + b_n)| = |(a - a_n) + (b - b_n)| < |a - a_n| + |b - b^n| < \epsilon$. This is just the desired assertion.
3. $|t_n - 0| = |t_n| = t_n \leq s_n = |s_n| = |s_n - 0|$. We are given that for any ϵ, there exists an N such that for $n > N$, $|s_n - 0| < \epsilon$. Since $|t_n - 0| \leq |s_n - 0|$ the same N serves for the sequence $\{t_n\}$.
4. We wish to prove that given ϵ, there is an N such that for $n > N\,|cs - cs_n| < \epsilon$. But $|cs - cs_n| = c\,|s - s_n|$. Given ϵ, choose the value ϵ/c. Then we know that there is an N such that for $n > N$, $|s - s_n| < \epsilon/c$. Then for the same n, $c\,|s - s_n| < \epsilon$ and $|cs - cs_n| < \epsilon$.
5. To prove that $|a + s - (a + s_n)| < \epsilon$ for all n greater than some N is the same as proving $|s - s_n| < \epsilon$ for all n greater than some N. But we are given the latter.
6. We know that given ϵ, there is an N such that for $n > N$, $|s - s_n| < \epsilon$. Now the even numbered terms are just those for which n is even. Hence these automatically satisfy the $\epsilon - N$ condition. To treat the even-numbered numbers as a separate sequence we have but to recognize that we replace s_2 by a_1, s_4 by a_2, etc.
7. Since $\lim\limits_{x\to a} f(x) = b$, we know that given any ϵ there is a δ such that when $0 < |x - a| < \delta$, then $|f(x) - b| < \epsilon$. Now $\{x_n\}$ is a sequence with limit a. Then there is some N such that for $n > N$, $|x_n - a| < \delta$ (the δ here serves as the ϵ of the sequence definition). But for $|x_n - a| < \delta$, $|f(x_n) - b| < \epsilon$. Hence, since ϵ was arbitrary to start with, we have shown that given ϵ, there is an N such that for $n > N$, $|f(x_n) - b| < \epsilon$ or $\{f(x_n)\}$ has the limit b.
8. Suppose the sequence $\{s_n\}$ has the limits a and b. Then given ϵ, there is an N_1 such that for $n > N_1$, $|a - s_n| < \epsilon$ and for the same ϵ, there is an N_2 such that for $n > N_2$, $|b - s_n| < \epsilon$. Now $|a - b| = |a - s_n + s_n - b| < |a - s_n| + |s_n - b|$. For n greater than N_1 and N_2, $|a - s_n| + |s_n - b| < 2\epsilon$. But this is true for any ϵ however small. If a and b are distinct numbers $a - b$ cannot be less than $2\,|b - a|/2$. Hence the sequence has only one limit.

CHAPTER 25, SECTION 8

1. The statement means that a certain family of sequences has a common limit. The n-th term, S_n, of any one sequence is formed by selecting a set of subintervals $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$ filling out the interval from 0 to 2, choosing an x_i in each subinterval, and forming $S_n = 3x_1^2\Delta x_1 + 3x_2^2\Delta x_2 + \ldots + 3x_n^2\Delta x_n$. As n increases the maximum subinterval must approach 0. Each $\{S_n\}$ has a limit and all have the same limit. This limit is the integral.
2. (a) The integral on the left is the limit of a sequence whose n-th tern is $S_n = [f(x_1) + g(x_1)]\Delta x_1 + [f(x_2) + g(x_2)]\Delta x_2 + \ldots + [f(x_n) + g(x_n)]\Delta x_n$ where the Δx's fill out the interval (a, b), x_1 is any value of x in Δx_1, etc., and as n becomes infinite the maximum Δx approaches 0. We can, as a matter of algebra write $S_n = S_n^{(1)} + S_n^{(2)}$ where $S_n^{(1)} = f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \ldots + f(x_n)\Delta x_n$ and $S_n^{(2)} = g(x_1)\Delta x_1 + g(x_2)\Delta x_2 + \ldots + g(x_n)\Delta x_n$.

   Moreover $\lim\limits_{n\to\infty} S_n^{(1)} = \int_a^b f(x)\,dx$ and $\lim\limits_{n\to\infty} S_n^{(2)} = \int_a^b g(x)\,dx$. But $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} S_n^{(1)} + \lim\limits_{n\to\infty} S_n^{(2)}$ because $\{S_n\}$ is a sum of $\{S_n^{(1)}\}$ and $\{S_n^{(2)}\}$ and the limit of a sum of two sequences is the sum of the limits.

   (b) The integral on the left is the limit of a sequence whose n-th term $S_n = cf(x_1)\Delta x_1 + cf(x_2)\Delta x_2 + \ldots cf(x_n)\Delta x_n$ with the usual understandings on the Δx's and the x_i's. If we factor out the c from each term then $S_n = cS_n^{(1)}$. Moreover $\lim\limits_{n\to\infty} S_n^{(1)} = \int_a^b f(x)\,dx$. Now $\lim\limits_{n\to\infty} S_n = c \lim\limits_{n\to\infty} S^{(1)}$ (see Section 3, Exercise 4). Hence $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$.
3. Let $f(x) = g(x) = x$. Then $\int_a^b f(x)g(x)\,dx = b^3/3 - a^3/3$. But $\int_a^b f(x)\,dx = b^2/2 - a^2/2$ and $\int_a^b g(x)\,dx = b^2/2 - a^2/2$. Then the integral of the product does not equal the product of the integrals. Since this is the case for at least one pair of functions, the equality cannot hold generally.

CHAPTER 25, SECTION 9

1. By definition (see (50)) $\int_0^2 f(x)\,dx = \int_0^1 x^2\,dx + \int_1^2 3x^2\,dx = 7\tfrac{1}{3}$.
2. (a) $\int_0^a dx/\sqrt{a^2 - x^2} = \lim\limits_{\epsilon\to 0} \int_0^{a-\epsilon} dx/\sqrt{a^2 - x^2} = \lim\limits_{\epsilon\to 0} \sin^{-1}(x/a)\Big|_0^{a-\epsilon} = \sin^{-1} 1 = \pi/2$.

   (b) $\int_0^1 dx/x^2 = \lim\limits_{\epsilon\to 0} \int_\epsilon^1 dx/x^2 = \lim\limits_{\epsilon\to 0} (1/\epsilon - 1)$. No value.

   (c) $\int_0^a x\,dx/\sqrt{a^2 - x^2} = \lim\limits_{\epsilon\to 0} \int_0^{a-\epsilon} x\,dx/\sqrt{a^2 - x^2} = \lim\limits_{\epsilon\to 0} (-\sqrt{a^2 - x^2})\Big|_0^{a-\epsilon} = a$.

   (d) $\int_0^2 dx/x = \lim\limits_{\epsilon\to 0} \int_\epsilon^2 dx/x = \lim\limits_{\epsilon\to 0} (\log 2 - \log\epsilon)$. No value.

(e) $\int_0^1 dx/x^a = \lim_{\epsilon\to 0} \int_\epsilon^1 dx/x^a = \lim_{\epsilon\to 0} x^{-a+1}/(1-a)\Big|_\epsilon^1 = 1/(1-a)$ for $0 < a < 1$, because for $0 < a < 1$, x^{-a+1} is a positive power of x.

(f) $\int_0^1 dx/x^a = \lim_{\epsilon\to 0} x^{-a+1}/(1-a)\Big|_\epsilon^1$. No value for $a > 1$ because for $a > 1$, x^{-a+1} is a negative power of x and becomes infinite as ϵ approaches 0.

(g) $\int_0^1 dx/(x-1)^{2/3} = \lim_{\epsilon\to 0} \int_0^{1-\epsilon} dx/(x-1)^{2/3} = \lim_{\epsilon\to 0} 3(x-1)^{1/3}\Big|_0^{1-\epsilon} = \lim_{\epsilon\to 0} -3\sqrt[3]{\epsilon} + 3 = 3.$

(h) $\int_1^2 dx/(x-1)^{2/3} = \lim_{\epsilon\to 0} \int_{1+\epsilon}^2 dx/(x-1)^{2/3} = \lim_{\epsilon\to 0} 3(x-1)^{1/3}\Big|_{1+\epsilon}^2 = \lim_{\epsilon\to 0} 3 - 3\sqrt[3]{\epsilon} = 3.$

(i) $\int_0^1 dx/(1-x)^2 = \lim_{\epsilon\to 0} \int_0^{1-\epsilon} dx/(1-x)^2 = \lim_{\epsilon\to 0} [1/(1-x)]\Big|_0^{1-\epsilon}$. No value.

(j) $\int_0^1 \cot x\, dx = \lim_{\epsilon\to 0} \int_\epsilon^1 \cot x\, dx = \lim_{\epsilon\to 0} \log|\sin x|\Big|_\epsilon^1$. No value.

(k) $\int_0^{\pi/2} \sec x\, dx = \lim_{\epsilon\to 0} \int_0^{\pi/2-\epsilon} \sec x\, dx = \lim_{\epsilon\to 0} \log|\sec x + \tan x|\Big|_0^{\pi/2-\epsilon}$. No value.

(l) $\int_{-1}^0 dx/(x^2-x) = \lim_{\epsilon\to 0} \int_{-1}^{-\epsilon} dx/(x^2-x) = \lim_{\epsilon\to 0} \int_{-1}^{-\epsilon} [(-1/x) + 1/(x-1)]dx = \lim_{\epsilon\to 0} [-\log|-\epsilon| + \log|-1| + \log|-\epsilon-1| - \log|-2|] = -\lim_{\epsilon\to 0} \log|\epsilon| - \log 2$. No value.

3. (a) $\int_{-1}^1 dx/x^2 = \lim_{\epsilon_1\to 0} \int_{\epsilon_1}^1 dx/x^2 + \lim_{\epsilon_2\to 0} \int_{-1}^{-\epsilon_2} dx/x^2 = \lim_{\epsilon_1\to 0} \{-(1/1) + 1/\epsilon_1\} + \lim_{\epsilon_2\to 0} \{1/\epsilon_2 + 1/1\}$. Hence no value.

(b) $\int_0^2 dx/(x-1)^2 = \lim_{\epsilon_1\to 0} \int_0^{1-\epsilon} dx/(x-1)^2 + \lim_{\epsilon_2\to 0} \int_{1+\epsilon_2}^2 dx/(x-1)^2$

$= \lim_{\epsilon_1\to 0} \{-(1/\epsilon_1) - 1\} + \lim_{\epsilon_2\to 0} \{-[1/(2-1)] + 1/\epsilon_2\}$. Hence no value.

(c) $\int_{-1}^1 dx/x^{2/3} = \lim_{\epsilon_1\to 0} \int_{\epsilon_1}^1 dx/x^{2/3} + \lim_{\epsilon_2\to 0} \int_{-1}^{-\epsilon_2} dx/x^{2/3} = \lim_{\epsilon_1\to 0} 3x^{1/3}\Big|_{\epsilon_1}^1 + \lim_{\epsilon_2\to 0} 3x^{1/3}\Big|_{-1}^{-\epsilon_2} = 6.$

(d) $\int_0^{3a} 2x\, dx/(x^2-a^2)^{2/3} = \lim_{\epsilon_1\to 0} \int_0^{a-\epsilon_1} 2x\, dx/(x^2-a^2)^{2/3} + \lim_{\epsilon_2\to 0} \int_{a+\epsilon_2}^{3a} 2x\, dx/(x^2-a^2)^{2/3} = \lim_{\epsilon_1\to 0} 3(x^2-a^2)^{1/3}\Big|_0^{a-\epsilon_1} + \lim_{\epsilon_2\to 0} 3(x^2-a^2)^{1/3}\Big|_{a+\epsilon_2}^{3a} = 3a^{2/3}.$

(e) $\int_1^4 dx/(x-3) = \lim_{\epsilon_1\to 0} \int_1^{3-\epsilon_1} dx/(x-3) + \lim_{\epsilon_2\to 0} \int_{3+\epsilon_2}^4 dx/(x-3) =$

$= \lim_{\epsilon_1\to 0} \log|x-3|\Big|_1^{3-\epsilon_1} + \lim_{\epsilon_2\to 0} \log|(x-3)|\Big|_{3+\epsilon_2}^4$. No value.

(f) $\int_1^4 dx/(x-3)^2 = \lim_{\epsilon_1 \to 0} \int_1^{3-\epsilon_1} dx/(x-3)^2 + \lim_{\epsilon_2 \to 0} \int_{3+\epsilon_2}^4 1/(x-3)^2 = \lim_{\epsilon_1 \to 0} \{1/\epsilon_1 - 2\}$ $+ \lim_{\epsilon_2 \to 0} \{-1 - 1/\epsilon_2\}$. Hence no value.

4. (a) $\int_0^\infty e^{-x}dx = \lim_{b\to\infty} \int_0^b e^{-x}dx = \lim_{b\to\infty} -e^{-x}\Big|_0^b = 1.$

(b) $\int_0^\infty e^{-ax}dx = \lim_{b\to\infty} \int_0^b e^{-ax}dx = \lim_{b\to\infty} \{-(1/a)e^{-ab} + 1/a\} = 1/a.$

(c) $\int_1^\infty dx/x = \lim_{b\to\infty} \int_1^b dx/x = \lim_{b\to\infty} \{\log b - \log 1\}$. No value.

(d) $\int_1^\infty dx/x^2 = \lim_{b\to\infty} \int_1^b dx/x^2 = \lim_{b\to\infty} \{1 - 1/b\} = 1.$

(e) $\int_1^\infty dx/x^a = \lim_{b\to\infty} \int_1^b dx/x^a = \lim_{b\to\infty} x^{1-a}/(1-a)\Big|_1^b = 1/(a-1)$ for $a > 1$.

(f) $\int_0^\infty x\,dx/(1+x^2) = \lim_{b\to\infty} \int_0^b x\,dx/(1+x^2) = \lim_{b\to\infty} \tfrac{1}{2}\log(1+x^2)\Big|_0^b$. No value.

(g) $\int_0^\infty \cos x\,dx = \lim_{b\to\infty} \int_0^b \cos x\,dx = \lim_{b\to\infty} \sin x\Big|_0^b$. Since $\sin b$ keeps oscillating in value between -1 and 1 as b become infinite, there is no limit.

(h) $\int_2^\infty dx/(x-1)^2 = \lim_{b\to\infty} \int_2^b dx/(x-1)^2 = \lim_{b\to\infty} -[1/(x-1)]\Big|_2^b = 1.$

(i) $\int_1^\infty \log x\,dx = \lim_{b\to\infty} \int_1^b \log x\,dx = \lim_{b\to\infty} (x\log x - x)\Big|_1^b = \lim_{b\to\infty} x(\log x - 1)\Big|_1^b$. Hence no value.

(j) $\int_0^\infty xe^{-x}dx = \lim_{b\to\infty} \int_0^b xe^{-x}dx = \lim_{b\to\infty} \{-xe^{-x} - e^{-x}\}\Big|_0^b = 1.$

(k) $\int_{-\infty}^\infty dx/(1+x^2) = \lim_{b\to\infty} \int_0^b dx/(1+x^2) + \lim_{a\to-\infty} \int_a^0 dx/(1+x^2) = \lim_{b\to\infty}$ $\{\tan^{-1} b - \tan^{-1} 0\} + \lim_{a\to-\infty} \{\tan^{-1} 0 - \tan^{-1} a\} = \pi/2 + \pi/2 = \pi.$

(l) $\int_0^\infty xe^{-x^2}dx = \lim_{b\to\infty} \int_0^b xe^{-x^2}dx = \lim_{b\to\infty} -\tfrac{1}{2}e^{-x^2}\Big|_0^b = \tfrac{1}{2}.$

(m) $\int_2^\infty dx/(x^2 - x) = \lim_{b\to\infty} \int_2^b \{(-1/x) + 1/(x-1)\}\,dx$
$= \lim_{b\to\infty} \{-\log b + \log 2 + \log(b-1)\} = \log 2 + \lim_{b\to\infty} \log(b-1)/b$
$= \log 2 + \lim_{b\to\infty} \log(1 - 1/b) = \log 2.$

(n) $\int_0^\infty e^{-x}\cos x\,dx = \lim_{b\to\infty} \int_0^b e^{-x}\cos x\,dx = \lim_{b\to\infty} e^{-x}(\sin x - \cos x)/2\Big|_0^b$. Since $\sin x$ and $\cos x$ oscillate between -1 and 1 whereas e^{-b} approaches 0, $e^{-b}(\sin b - \cos b)$ approaches 0. Hence the value is $\tfrac{1}{2}$ because of the contribution from $x = 0$.

5. $\int_0^\infty 100e^{-kt}\,dt = \lim_{b\to\infty} \int_0^b 100e^{-kt}\,dt = \lim_{b\to\infty} -(100/k)e^{-kt}\Big|_0^b = 100/k.$

6. $\int_0^\infty 100\,dt/(1+100kt) = \lim_{b\to\infty} \int_0^b 100\,dt/(1+100kt) = \lim_{b\to\infty} (1/k)\log(1+100kt)\Big|_0^b$ $= \infty.$

7. $\int_0^\infty (32/k)(1-e^{-kt})\,dt = \lim_{b\to\infty} \int_0^b (32/k)(1-e^{-kt})\,dt = \lim_{b\to\infty} (32/k)(t+(1/k)e^{-kt})\Big|_0^b = \infty.$

8. (a) Formula (36) of Chapter 16 reads $F_i = GMmt2\pi x_i \cos\theta_i \Delta x/(PS)^2$. It gives the force which a small ring of inner radius x_i and width Δx exerts on a mass m, h units above the center of the ring. If we regard the plane as a "sum" of small rings with radii extending from 0 to ∞ we get for the total force exerted by the infinite plane $F = \int_0^\infty 2\pi GMmt(x\cos\theta/PS^2)\,dx$. Before integrating we can convert all variables to a function of x. Since $\cos\theta = h/\sqrt{x^2+h^2}$ and $PS^2 = x^2+h^2$ we have $F = 2\pi GMmth\int_0^\infty [x/(x^2+h^2)^{3/2}]\,dx$. The integral is readily evaluated by letting $u = x^2+h^2$. The result is $2\pi GMmt$. This conclusion could also be derived from (41) by letting a become infinite.

   (b) The plane is infinite in extent and, loosely speaking, the distance from m to the plane, that is, the distance h, is negligible. Or, one can say, from a point on the plane far away the distance h is insignificant compared to the attracting matter.

9. $s = \int_0^{-\infty} \sqrt{\rho^2+\rho'^2}\,d\theta = \int_0^{-\infty} \sqrt{a^2e^{2b\theta}+a^2b^2e^{2b\theta}}\,d\theta = \int_0^{-\infty} \sqrt{a^2+a^2b^2}\,e^{b\theta}\,d\theta$

   $= (\sqrt{a^2+a^2b^2}/b)e^{b\theta}\Big|_0^{-\infty} = -\sqrt{a^2+a^2b^2}/b$. The minus enters because we go from 0 to $-\infty$ whereas we took $ds/d\theta$ positive. To find the length of the tangent PQ note that ψ, the angle between the radius vector and the tangent at $\theta = 0$ is also the inclination of the tangent in the usual rectangular coordinate sense. Now $\tan\psi = \rho/\dot\rho = 1/b$. Hence the equation of the tangent is $y - 0 = (1/b)(x-a)$ and since at Q, $x = 0$, y at $Q = -a/b$. Now find the distance from $(a, 0)$ to $(0, -a/b)$. This proves to be the length of arc just computed.

CHAPTER 25, SECTION 10

1. (a) In $0 \le x \le \pi/2$, we have $0 \le \sin x \le 1$; hence $0 \le \sin^5 x \le 1$. Thus lower and upper bounds are $0(\pi/2 - 0)$ and $1(\pi/2 - 0)$ or 0 and $\pi/2$.

   (b) In $0 \le x \le \pi$, we have $-1 \le \sin x \le 1$, hence $-1 \le \sin^5 x \le 1$. Thus bounds are $-1(\pi - 0)$ and $1(\pi - 0)$ or $-\pi$ and π.

   (c) In $0 \le x \le 1$, we have $0 \le x^{10} + 9x^9 \le 1 + 9 = 10$. Thus the bounds are $0(1 - 0)$ and $10(1 - 0)$.

(d) In $0 < x \le 1$, we have $1 = e^0 \le e^{x^2} \le e^1 = e$. Thus the bounds are $1(1-0)$ and $e(1-0)$.

(e) In $0 \le x \le 1$, we have $e^{-1} \le e^{-x^2} \le e^0 = 1$. Thus the bounds are $e^{-1}(1-0)$ and $e^0(1-0)$.

2. $F(3) = \int_0^3 x\,dx = 9/2$.

3. Since the integrand is positive, we may think of $F(u)$ as the value of the area under a curve lying above the x-axis and between $x = 1$ and $x = u$. Thus as u increases a greater area is included and so $F(u)$ increases. An analytical argument may be based on the fact that by the fundamental theorem, $F'(u) = (u \sin u + e^u \sin u)^2 > 0$.

4. By the fundamental theorem, we obtain $F'(u) = u^{3/2} \sin u$. Thus $F'(u) = x^{3/2} \sin x$.

5. (a) By the mean value theorem $\int_\pi^{\pi+h} \sin x\,dx = [(\pi + h) - \pi] \sin c = h \sin c$ where $\pi \le c \le \pi + h$. Since $|\sin c| < 1$, we have $\left|\int_\pi^{\pi+h} \sin x\,dx\right| \le |h|$. Thus the required limit is 0.

(b) By the mean value theorem $\int_{\pi/2}^{\pi/2+h} \sqrt{\sin x}\,dx = \sqrt{\sin c}\,(\pi/2 + h - \pi/2)$. Since $\sqrt{\sin c} \le 1$, then $\left|\int_{\pi/2}^{\pi/2+h} \sqrt{\sin x}\,dx\right| \le |h|$. As h approaches 0 the integral must.

(c) Let $t^2 = u$. Then we want $(d/dt) \int_0^u \sin x\,dx = [(d/du) \int_0^u \sin x\,dx]\,(du/dt)$. By the fundamental theorem the latter equals $\sin u\; du/dt = 2t \sin t^2$.

(d) Assuming that $\sin x$ is positive for $0 \le x \le t^2$, we use the method of (c) to obtain the answer $2t\sqrt{\sin t^2}$.

(e) Assuming $f(t)$ is continuous for $a \le t \le a + h$ at least for some sufficiently small value of h, we use the mean value theorem to write $\int_a^{a+h} f(t)\,dt = [(a + h) - a]f(c) = hf(c)$ where $a \le c \le a + h$. Thus the required limit reduces to $\lim_{h\to 0} f(c)$ where $a \le c \le a + h$. By the assumed continuity of f, this limit is $f(a)$.

(f) Use the result of (e) with $h = x$ and $a = 0$ to obtain $\sqrt{0^3 + 4} = 2$.

(g) Use the mean value theorem to write the given limit in the form $\lim_{x\to 1} \tan^{-1} c$ where $1 \le c \le x$. Thus the result is $\tan^{-1} 1 = \pi/4$.

6. (a) $F(0) = \int_0^0 dt/\sqrt{t^3+1} = 0$.

   (b) $F(-\frac{1}{2}) = \int_0^{-1/2} dt/\sqrt{t^3+1} = -\int_{-1/2}^{0} dt/\sqrt{t^3+1}$. Since the integrand is positive the integral must be and so we conclude that $F(-\frac{1}{2}) < 0$.

   (c) In order for the integrand to be real, we must have $t^3 + 1 > 0$ in the interval 0 to -3 for t. Hence $F(-3)$ does not exist.

   (d) By the fundamental theorem, $F'(x) = 1/\sqrt{x^3+1}$.

   (e) Differentiating the result of (d), we have $F''(x) = (-\frac{3}{2})x^2/\sqrt{(x^3+1)^3}$.

   (f) Taylor's theorem to four terms calls for $F(x) = F(0) + F'(0)x + F''(0)x^2/2! + F'''(\mu)x^3/3!$ where $0 \leq \mu \leq x$. Using the values of $F(0)$, $F'(0)$ and $F''(0)$ already obtained we get the text result.

   (g) Since $F'(x) > 0$ (see (d)) for $x > -1$, $F(x)$ is, indeed, increasing in this domain.

7. (a) By the fundamental theorem, $F'(x) = (x+1)\tan^{-1}[1/(1+x^2)]$.

   (b) Find $F'(x)$ from (a). Then $F'(0) = \tan^{-1} 1 = \pi/4$.

8. As in Exercise 7, we obtain, $F'(x) = e^{-x^2}$. Thus $F'(1) = e^{-1}$.

9. As pointed out in the suggestion the attraction at a point h units from the center is $(G/h^2)\int_0^h 4\pi\rho(r)r^2\,dr$. The attraction that the sphere of radius a and mean density $\bar{\rho}$ exerts at the surface is $\frac{4}{3}\pi Ga^3\bar{\rho}/a^2$. Hence $(G/h^2)\int_0^h 4\pi\rho(r)r^2\,dr = \frac{4}{3}\pi a\bar{\rho}G$. Then $\int_0^h \rho(r)r^2\,dr = h^2(a\bar{\rho}/3)$. Differentiate both sides with respect to h. Then $\rho(h)h^2 = 2h(2\bar{\rho}/3)$. Hence $\rho(h) = 2a\bar{\rho}/3h$ where h is distance from the center of the earth.